Applied Mathematics

A Comprehensive Course for Leaving Certificate 3rd Edition

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Errata

Chapter 4 Answers (p.286):

Exercise 4A: ~~14)~~ 16) 120 m. ~~15)~~ 17) 23 m.

Exercise 4B: 16) (i) ~~14.9°~~ 12.6°, 86.4°,

Chapter 5

Example 5A2 solution (p. 78): $\Rightarrow v = \sout{18.3}\ 57.9 \text{ m s}^{-1}$

~~(iv)~~ (v) $\text{K.E.} = \frac{1}{2}mv^2 = \frac{1}{2}(25)(4.21)^2 = 221.6 \text{ J}$

Example 5B5 solution (p.82): ~~(v)~~ (iv) $\text{P.E.} = mgh = 25g\left(4.21\sin 30^\circ\right) = 515.7 \text{ J}$

Chapter 6

Example 6E3:

Alternative method for end of (i) using dot product

If $\vec{u_1} \perp \vec{v_1} \Leftrightarrow \vec{u_1} \cdot \vec{v_1} = 0$

$$\Rightarrow \left(u\cos\alpha\vec{i} + u\sin\alpha\vec{j}\right) \cdot \left(\frac{u\cos\alpha(1-3e)}{4}\vec{i} + u\sin\alpha\vec{j}\right) = 0$$

$$\Rightarrow \frac{u^2\cos^2\alpha(1-3e)}{4} + u^2\sin^2\alpha = 0$$

$$\Rightarrow \cos^2\alpha(1-3e) + 4\sin^2\alpha = 0$$

$$\Rightarrow 4\tan^2\alpha = 3e - 1$$

$$\Rightarrow \tan\alpha = \sqrt{\frac{3e-1}{4}}$$

Exercise 6E (p.108): Q15 (i) the velocities of the two spheres ~~after~~ before impact,

Chapter 6 Answers (p.287):

Exercise 6E: 13) ~~$\frac{5}{3}$ at 36.9° N of W, 3.07 at 77.5° N of E.~~ 3.8 m s^{-1} @ 58° N of E, 2.1 m s^{-1} @ 62° N of W

Chapter 7

Exercise 7D (p. 128): Q8 (ii) ~~$\omega = \sqrt{\frac{mg}{h}}$~~ $\omega = \sqrt{\frac{g}{h}}$

Extra question – Q11

(i) Derive an expression for the work done when a spring of elastic constant k N m^{-1} is stretched by x m.

(ii) A particle of mass 5 kg is attached to one end of a bungee cord of natural length 2 m and elastic constant 25 N m^{-1}. The other end of the bungee cord is attached to a point Q. The 5 kg particle is held next to point Q, released and allowed to fall vertically. Using conservation of energy, find the extension of the bungee cord when the 5 kg particle next comes to rest.

Chapter 7 Answers (p.288):

Exercise 7D: Q9 (ii) ~~$\frac{3g}{2}$ m s^{-2}~~ $\frac{g}{2}$ m s^{-2}

Q11) (i) $\frac{1}{2}kx^2$, (ii) 5.38 m.

Chapter 12

Example 12C2 (p.209)

The alternative solution to the right has been added to this example.

Alternative solution: use $u_n = C(k^n) + D$

$$2 = C(3^1) + D, \quad \Rightarrow 2 = 3C + D \quad \mathbf{C}$$

$$6 = C(3^2) + B, \quad \Rightarrow 6 = 9C + D \quad \mathbf{D}$$

$\mathbf{D} - \mathbf{C}$: $4 = 6C$, $\Rightarrow C = \frac{2}{3}$ and then $D = 0$

$\Rightarrow$ the solution is: $u_n = \frac{2}{3}(3^n) = 2(3^{n-1})$

It is a matter of preference whether to use $n-1$ or n as the index in the general solution.

Example 12D1 (ps.212/3)

The alternative solution to the right has been added to this example.

Alternative solution: use $u_n = C(1.004^n) + D$

$$1600 = C + D \quad \mathbf{C}$$

$$1706.4 = 1.004C + D \quad \mathbf{D}$$

$\mathbf{D} - \mathbf{C}$: $0.004C = 106.4$, $\Rightarrow C = 26{,}600$

In $\mathbf{C}$: $D = 1600 - 26600 = -25{,}000$

$\Rightarrow$ the solution is: $u_n = 26{,}600(1.004^n) - 25{,}000$

Example 12E2(a) (p.216): $u_1 = \cancel{3}$

There are calculation errors in examples 12F2, 12F3 & 12F4. The corrected versions are below, with all corrections circled in red.

Example 12F2 (p.219)

(a) Solve the inhomogeneous first order difference equation $u_{n+1} = n^2 - 2n + 1 - 3u_n$ with $u_1 = 0$.

(b) Hence find u_9.

Solution: (a) Rewrite it with all terms of form u_n on the left: $u_{n+1} + 3u_n = n^2 - 2n + 1$

The particular solution is of the form: $u_n = a + bn + cn^2$.

The general solution of the difference equation is of the form: $u_n = A(-3)^{n-1}$.

Putting these together gives: $u_n = A(-3)^{n-1} + a + bn + cn^2$.

As there are four unknowns still here, A, a, b and c, we will need the first four terms of the sequence.

$u_1 = 0$ and $u_{n+1} = n^2 - 2n + 1 - 3u_n$

$\Rightarrow u_2 = 1^2 - 2(1) + 1 - 3(0) = \textcircled{0}$ $\{n = 1\}$

& $u_3 = 2^2 - 2(2) + 1 - 3(0) = 1$ $\{n = 2\}$

& $u_4 = 3^2 - 2(3) + 1 - 3(1) = \textcircled{1}$ $\{n = 3\}$

Now use these in our solution to find the four unknowns.

$u_1 = A(-3)^{1-1} + a + b(1) + c(1)^2 = 0$, $\Rightarrow A + a + b + c = 0$ **A**

$u_2 = A(-3)^{2-1} + a + b(2) + c(2)^2 = \textcircled{0}$, $\Rightarrow -3A + a + 2b + 4c = \textcircled{0}$ **B**

$u_3 = A(-3)^{3-1} + a + b(3) + c(3)^2 = 1$, $\Rightarrow 9A + a + 3b + 9c = 1$ **C**

$u_4 = A(-3)^{4-1} + a + b(4) + c(4)^2 = \textcircled{1}$, $\Rightarrow -27A + a + 4b + 16c = \textcircled{1}$ **D**

$(-9 \times \mathbf{A}) + \mathbf{C}$: $-8a - 6b = 1$, $\Rightarrow 8a + 6b = -1$ **E**

$(3 \times \mathbf{B}) + \mathbf{C}$: $4a + 9b + 21c = 1$ **F**

$(3 \times \mathbf{C}) + \mathbf{D}$: $4a + \textcircled{13}b + 43c = \textcircled{4}$ **G**

$(43 \times \mathbf{F}) + (-21 \times \mathbf{G})$: $88a + 114b = \textcircled{-41}$ **H**

$(-11 \times \mathbf{E}) + \mathbf{H}$: $48b = \textcircled{-30}$, $\Rightarrow b = -\frac{\textcircled{5}}{8}$

$\Rightarrow$ in **E**: $8a + 6\left(\frac{-5}{\textcircled{8}}\right) = -1$, $\Rightarrow a = \frac{\textcircled{11}}{32}$

$\Rightarrow$ in **F**: $4\left(\frac{11}{32}\right)+9\left(\frac{-5}{8}\right)+21c=1,$ $\quad\Rightarrow c=\frac{1}{4}$

$\Rightarrow$ in **A**: $A+\frac{11}{32}-\frac{5}{8}+\frac{1}{4}=0,$ $\quad\Rightarrow A=\frac{1}{32}$

$$\Rightarrow u_n=\frac{1}{32}(-3)^{n-1}+\frac{11}{32}-\frac{5n}{8}+\frac{n^2}{4}.$$

$$\text{(b)}\quad u_9=\frac{1}{32}(-3)^{9-1}+\frac{11}{32}-\frac{5(9)}{8}+\frac{9^2}{4}=220$$

Example 12F3 (p.220)

(a) Solve the inhomogeneous second order difference equation $u_{n+1}-3u_n-4u_{n-1}=1-3n$ with $u_1=1$ and $u_2=2$.

(b) Hence find u_{10}.

Solution: (a) From the table above the particular solution is of the form: $u_n = a + bn$.

The characteristic equation for the corresponding inhomogeneous difference equation is: $x^2-3x-4=0$.

Solving this: $(x+1)(x-4)=0,$ $\quad\Rightarrow x=-1$ or 4

The general solution of the difference equation is of the form: $u_n=A(-1)^n+B(4)^n$.

Putting these together gives: $u_n=A(-1)^n+B(4)^n+a+bn.$

As there are four unknowns still here, A, B, a and b, we will need the first four terms of the sequence.

$$u_1=1,\ u_2=2\quad\text{and}\quad u_{n+1}-3u_n-4u_{n-1}=1-3n$$

$$\Rightarrow u_3-3(2)-4(1)=1-3(2),\qquad\Rightarrow u_3=5$$

$$\&\ u_4-3(5)-4(2)=1-3(3),\qquad\Rightarrow u_4=15$$

Now use these in our solution to find the four unknowns.

$u_1=A(-1)^1+B(4)^1+a+b(1)=1,$ $\quad\Rightarrow -A+4B+a+b=1$ **A**

$u_2=A(-1)^2+B(4)^2+a+b(2)=2,$ $\quad\Rightarrow A+16B+a+2b=2$ **B**

$u_3=A(-1)^3+B(4)^3+a+b(3)=5,$ $\quad\Rightarrow -A+64B+a+3b=5$ **C**

$u_4=A(-1)^4+B(4)^4+a+b(4)=15,$ $\quad\Rightarrow A+256B+a+4b=15$ **D**

D $-$ **B**: $240B+2b=13$ **E**

C $-$ **A**: $60B+2b=4$ **F**

E $-$ **F**: $180B=9,$ $\quad\Rightarrow B=\frac{1}{20}=0.05$

$\Rightarrow$ in **E**: $240\left(\frac{1}{20}\right)+2b=13,$ $\quad\Rightarrow b=\frac{1}{2}$

$\Rightarrow$ in **A**: $-A+\frac{1}{5}+a+\frac{1}{2}=1,$ $\quad\Rightarrow -A+a=\frac{3}{10}$ **G**

$\Rightarrow$ in **B**: $A+\frac{4}{5}+a+1=2,$ $\quad\Rightarrow A+a=\frac{1}{5}$ **H**

G $+$ **H**: $2a=0.5,$ $\quad\Rightarrow a=0.25$

$\Rightarrow$ in **H**: $A+0.25=\frac{1}{5},$ $\quad\Rightarrow A=-0.05,$ $\quad\Rightarrow u_n=-0.05(-1)^n+0.05(4)^n+0.25+0.5n.$

(b) $u_{10}=-0.05(-1)^{10}+0.05(4)^{10}+0.25+0.5(10)=52434.$

Example 12F4 (p.221)

(a) Solve the difference equation $u_{n+2} = u_{n+1} + 6u_n + 2^n$ with $u_0 = 2$ and $u_1 = 3$.

(b) Hence find u_{10}.

Solution: (a) First of all need to have the difference equation in the right form: $u_{n+2} - u_{n+1} - 6u_n = 2^n$

The particular solution is of the form: $u_n = a + b.2^n$.

The characteristic equation for the corresponding inhomogeneous difference equation is: $x^2 - x - 6 = 0$.

Solving this: $(x+2)(x-3) = 0, \quad \Rightarrow x = -2 \text{ or } 3$

The general solution of the difference equation is of the form: $u_n = A(-2)^n + B(3)^n$.

Putting these together gives: $u_n = A(-2)^n + B(3)^n + a + b.2^n$

As there are four unknowns still here, *A*, *B*, *a* and *b*, we will need the first four terms of the sequence.

$$u_0 = 2,\ u_1 = 3 \quad \text{and} \quad u_{n+2} = u_{n+1} + 6u_n + 2^n$$

$$\Rightarrow u_2 = 3 + 6(2) + 2^0, \qquad \Rightarrow u_2 = 16$$

$$\& \ u_3 = 16 + 6(3) + 2^1, \qquad \Rightarrow u_3 = 36$$

Now use these in our solution to find the three unknowns.

$u_0 = A(-2)^0 + B(3)^0 + a + b.2^0 = 2, \qquad \Rightarrow A + B + a + b = 2$ **A**

$u_1 = A(-2)^1 + B(3)^1 + a + b.2^1 = 3, \qquad \Rightarrow -2A + 3B + a + 2b = 3$ **B**

$u_2 = A(-2)^2 + B(3)^2 + a + b.2^2 = 16, \qquad \Rightarrow 4A + 9B + a + 4b = 16$ **C**

$u_3 = A(-2)^3 + B(3)^3 + a + b.2^3 = 36, \qquad \Rightarrow -8A + 27B + a + 8b = 36$ **D**

$(2\times\mathbf{A}) + \mathbf{B}$: $5B + 3a + 4b = 7$ **E**

$(2\times\mathbf{B}) + \mathbf{C}$: $15B + 3a + 8b = 22$ **F**

$(2\times\mathbf{C}) + \mathbf{D}$: $45B + 3a + 16b = 68$ **G**

$\mathbf{F} - \mathbf{E}$: $10B + 4b = 15$ **H**

$\mathbf{G} - \mathbf{F}$: $30B + 8b = 46, \qquad \Rightarrow 15B + 4b = 23$ **I**

$\mathbf{I} - \mathbf{H}$: $5B = 8, \qquad \Rightarrow B = 1.6$

$\Rightarrow$ in **H**: $4b = 15 - 10(1.6), \qquad \Rightarrow b = -0.25$

$\Rightarrow$ in **E**: $3a = 7 - 5(1.6) - 4(-0.25), \qquad \Rightarrow a = 0$

$\Rightarrow$ in **A**: $A + 1.6 - 0.25 = 2, \qquad \Rightarrow A = 0.65$

$$\Rightarrow u_n = 0.65(-2)^n + 1.6(3)^n - 0.25(2^n).$$

$$\text{(b)} \quad u_{10} = 0.65(-2)^{10} + 1.6(3)^{10} - 0.25(2^{10}) = 94{,}888.$$

Exercise 12F(p.223): Q14(e) ~~8~~ 6 years?

Chapter 12 Answers

Exercise 12A (p.292): Q14 ~~30~~ m 80 m

Exercise 12B (p.293): Q3(c) $T_{n+1} = T_n + 2n + 1,\ T_1 = 2$ or $T_{n+2} = 2T_{n+1} - T_n + 2,\ T_1 = 2,\ T_2 = 5$

Q5(c) ~~–560~~ –66

Q11~~(a) ¼, ⅖, ¼, ⅖, ¼, ⅖, (b) ⅖~~ (a) ¼, ⅗, ¼, ⅗, ¼, ⅗, (b) ⅗.

Q13) …, 13, 25, 49, ~~87~~. 94

Exercise 12C (p.293): Q12(f) ~~8~~ 9 weeks, (k) ~~2890~~ 2745 kg.

Exercise 12D (p.293): Q6(c) ~~€318~~ €316
Q8(c) ~~€105.52.~~ €211.05

Exercise 12E (p.293): Q1(d) ~~$x^2+7x+10=0$~~ $x^2-7x+10=0$
Q6 ~~(a) $T_n=3^n+\frac{1}{5}(-5)^n$, (b) $-9{,}588{,}478$~~ (a) $T_n=\frac{19}{24}(3^n)+\frac{11}{40}(-5)^n$, (b) $-13{,}287{,}493$

Exercise 12F (p.294): Q15 (a) $M_n=447.06\left(\frac{11}{10}\right)^n-29.213\left(-\frac{10}{11}\right)^n-67.846-10.476n$, (b) 976, not quite
~~(a) $M_n=\frac{1}{91}\left[42000\left(\frac{11}{10}\right)^n-2662\left(-\frac{10}{11}\right)^n-7488-1092n\right]$, (b) 984, not quite.~~

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Example 13C2 (p.227): ~~$=\frac{5}{2}+\sqrt{3}=4.23$~~ $=\frac{5}{2}-\sqrt{3}=0.770$

Example 13E2(b) (p.233): there are a number of sign errors in the solution, all corrections circled in red.

(b) $\int e^x\cos x\,dx$

Let $u=\cos x$, $\Rightarrow \frac{du}{dx}=-\sin x$, $\Rightarrow du=-\sin x\,dx$

and let $dv=e^x\,dx$, $\Rightarrow \frac{dv}{dx}=e^x$, $\Rightarrow v=e^x$

$$\int u\,dv=uv-\int v\,du$$

$$\Rightarrow \int e^x\cos x\,dx=e^x\cos x+\int e^x\sin x\,dx$$

To integrate $\int e^x\sin x\,dx$ we do another integration by parts.

Let $u=\sin x$, $\Rightarrow \frac{du}{dx}=\cos x$, $\Rightarrow du=\cos x\,dx$

and let $dv=e^x\,dx$, $\Rightarrow \frac{dv}{dx}=e^x$, $\Rightarrow v=e^x$

$$\int u\,dv=uv-\int v\,du$$

$$\Rightarrow \int e^x\sin x\,dx=e^x\sin x-\int e^x\cos x\,dx$$

$$\Rightarrow \int e^x\cos x\,dx=e^x\cos x+\left(e^x\sin x-\int e^x\cos x\,dx\right)=e^x\cos x+e^x\sin x-\int e^x\cos x\,dx$$

$$\Rightarrow 2\int e^x\cos x\,dx=e^x\cos x+e^x\sin x+C$$

$$\Rightarrow \int e^x\cos x\,dx=\frac{e^x}{2}(\cos x+\sin x)+D$$

Example 13F3(b) (p.236): small error in calculation, all corrections ringed in red.

(b) In this example the quadratic expression in the denominator has a repeated root. To deal with this in partial fractions one of the fractions has this root normally, and the other has it squared.

$$\frac{2x+3}{x^2-6x+9}=\frac{2x+3}{(x-3)^2}=\frac{A}{x-3}+\frac{B}{(x-3)^2}$$

$$\Rightarrow \frac{2x+3}{(x-3)^2}=\frac{A(x-3)+B}{(x-3)^2}=\frac{Ax-3A+B}{(x-3)^2}$$

$\Rightarrow A=2$ and $-3A+B=3$ $\Rightarrow B=9$

$$\Rightarrow \frac{2x+3}{x^2-6x+9}=\frac{2}{x-3}+\frac{9}{(x-3)^2}$$

$$\int\frac{2x+3}{x^2-6x+9}\,dx=\int\left(\frac{2}{x-3}+9(x-3)^{-2}\right)dx$$

$$=2\ln(x-3)+\frac{9(x-3)^{-1}}{-1}+C=2\ln(x-3)-\frac{9}{(x-3)}+C$$

Chapter 13 Answers

Exercise 13A (p.294): Q4(b) $\dfrac{3\sin x^2 - 2x(3x-4)\cos x^2 \ln(3x-4)}{\sin^2 x^2 (3x-4)}$

Q4(c) $-4\sqrt{3x}\cos 2x + \dfrac{\sqrt{3}\cos^2 2x}{2\sqrt{x}}$

Q4(d) $-\dfrac{2e^{1-4x}}{\sqrt[3]{x^{10}}}\left(4\sqrt[3]{x^5} + \dfrac{5\sqrt[3]{x^2}}{3}\right)$

Q6 $-\dfrac{19}{16}$

Q9(b) $e^{\sin^2 x}\sin 2x$

Q9(c) $\dfrac{-2x}{\sqrt{(1-2x^2)^3}}\sin\left(\dfrac{1}{\sqrt{1-2x^2}}\right)$

Exercise 13C (p.294): Q1(e) $2x + 3\ln|\sec x| + C$

Exercise 13D (p.295): Q3(c) $\dfrac{(3e^x-4)^6}{9} + C$

Q4(b) $\dfrac{3}{2x} + \dfrac{\ln x}{2} + C$

Q4(c) $-2x - 8\ln(4-x) + D$

Q6(b) ~~$e^x - 6\ln(e^x+6) + C$~~ $\ln(e^x+6) + C$

Q6(f) ~~$-\dfrac{\sqrt{e^{6-3x^2}}}{3} + C$~~ $-\sqrt{e^{2-x^2}} + C$

Exercise 13E (p.295): Q2 ~~$\dfrac{\cos x(2\cos^2 x - 3\sin^2 x)}{3} + C$~~ $\dfrac{\cos x(\cos^2 x - 3)}{3} + C$

Q4(a) $\dfrac{e^{2x}(2x^2 - 2x + 1)}{4} + C$

Q4(c) $\dfrac{x^2\sin 4x}{4} + \dfrac{x\cos 4x}{8} - \dfrac{\sin 4x}{32} + C$

Q5(g) ~~−0.830~~ −0.793

Exercise 13F (p.295): Q5(a) $\dfrac{x^2}{2} + 7x + 17\ln(x-2) + C$

Q5(c) $x + \ln(x^2+1) - 9\tan^{-1} x + C$

Q9(g) ~~−1.15~~ −0.901

Q11(b) ~~0.123~~ 0.0512

Q11(c) ~~2~~ 13120

Q11(d) ~~905.6~~ 908.8

Q11(h) ~~18.9~~ 17.0

Q11(k) ~~2.57~~ 0.507

Chapter 14

Exercise 14A (p.242): Q13: ~~27 m s^{-1}~~ 31.5 m s^{-1}
Q14(a) & (b): ~~maximum~~ minimum

Exercise 14B (p.247): Q9: ~~$v = 3 + 4t$~~ $v = 3 + 3t$

Example 14C1(a) (p.248): Error in calculation in the solution.

(a) $\vec{v} = \frac{1}{2}t^2\vec{i} - 3t\vec{j}$, $\Rightarrow$ the position vector $\vec{r} = \int \vec{v}\, dt = \int\left(\frac{1}{2}t^2\vec{i} - 3t\vec{j}\right) dt = \frac{t^3}{6}\vec{i} - \frac{3t^2}{2}\vec{j} + \vec{C}$

When $t = 0$, $\vec{r} = 2\vec{i} + 18\vec{j}$ m, $\Rightarrow \vec{r} = \frac{0^3}{6}\vec{i} - \frac{3(0)^2}{2}\vec{j} + \vec{C} = 2\vec{i} + 18\vec{j}$, $\Rightarrow \vec{C} = 2\vec{i} + 18\vec{j}$

$$\Rightarrow \vec{r} = \frac{t^3}{6}\vec{i} - \frac{3t^2}{2}\vec{j} + 2\vec{i} + 18\vec{j} = \left(2 + \frac{t^3}{6}\right)\vec{i} + \left(18 - \frac{3t^2}{2}\right)\vec{j} \text{ m}$$

When $t = 4$, $\vec{r} = \left(2 + \frac{4^3}{6}\right)\vec{i} + \left(18 - \frac{3(4)^2}{2}\right)\vec{j} = \frac{38}{3}\vec{i} - 6\vec{j}$ m

$$\Rightarrow |\vec{r}| = \sqrt{\left(\frac{38}{3}\right)^2 + (-6)^2} = 14.0 \text{ m} \text{ and } \tan\alpha = \frac{6}{\frac{38}{3}} = \frac{18}{38} = \frac{9}{19}, \Rightarrow \alpha = 25.3^\circ \text{ below horizontal.}$$

Exercise 14C (p.252): Q12(a) & (b): ~~acceleration~~ velocity
Q15(b): ~~$(160\vec{i} + 160\vec{j})$ m~~ $(120\vec{i} + 80\vec{j})$ m s^{-1}

Extra question – Q16
A particle is moving in a horizontal circle of centre O, with radius r and constant angular speed ω.
(a) Show that the displacement of the particle relative to O at any time t is $\vec{s} = r\cos\omega t\vec{i} + r\sin\omega t\vec{j}$ m. Note that at time t = 0, $\vec{s}$ is along the $\vec{i}$ axis.
(b) Derive an expression for $\vec{v}$, the velocity of the particle at any time t.
(c) Use a dot product calculation to show that the particle's velocity and displacement are always perpendicular to each other.
(d) Show that the acceleration of the particle is always directed towards the centre O.

Chapter 14 Answers

Exercise 14A (p.296): Q4(d) ~~4 m s^{-2}~~ 7 m s^{-2}
Q5(b) ~~5 m s^{-1}~~ 7 m s^{-1}
Q12(c) ~~32.3 m s^{-1}~~ 41.3 m s^{-1} Q12(e) ~~95.3 m~~ 114 m
Q14(b) −0.135 m s^{-1}
Q16(d) ~~24.6~~ 24.1
Q19(b) ~~20.6 m~~ 33.1 m
Q21(b) ~~74.4 m~~ 71.9 m

Exercise 14B (p.296): Q5(d) ~~49.4 m~~ 35.1 m
Q9(d) ~~21.25 m~~ 20.75 m
Q13(b) ~~5 s, −2.8 m s^{-2}~~ 0 s, 8 m s^{-2}
Q14(g) ~~10 m~~ 11.3 m

Exercise 14C (p.296): Q9(a) ~~8.48 m, 51.0° N of W~~ 22.3 m, 17.2° N of W Q9(c) ~~113.5°~~ 175.4°
Q12(a) ~~10 m s^{-2}~~ 5.83 m s^{-2} Q12(c) 11.8 m ~~3.44 m~~
Q13(a) $\left(\frac{2}{3}t^3 - 2t + 6\right)\vec{i} + \left(2 - 2.5t^2\right)\vec{j}$ m, Q13(b) 4 m ~~7 m~~
Q15(c) ~~379.5 m s^{-1}~~ 301.3 m s^{-1}
Q16(b) $\vec{v} = -r\omega\sin\omega t\vec{i} + r\omega\cos\omega t\vec{j}$ m s^{-1}.

Chapter 15 Answers

Exercise 15A (p.296): Q8(j) $y = \frac{4^{12} e^{\frac{x}{2}(x+4)}}{(x+4)^{12}} = e^{\frac{x}{2}(x+4)+12\ln\left(\frac{4}{x+4}\right)}$

Q8(m) ~~$y = x + 2\ln\left(\frac{4}{x^2+4}\right) + \tan^{-1}\frac{x}{4} + 2$~~ $y = x + 3\ln\left(\frac{x^2+4}{4}\right) - 4\tan^{-1}\frac{x}{2} + 2$

Exercise 15F (p.297): Answers to questions 4, 5 and 6 are numbered in the wrong order. Q4 is actually Q5, Q5 is actually Q6 and Q6 is actually Q4.

Appendix 1
Dimensional Analysis

Using Dimensional Analysis to Verify Equations

Dimensional Analysis is a technique used to verify equations or formulae by checking that the dimensions (units) are consistent across all terms in the equation or formula.

Quantity	Symbol(s)	Dimension (Unit)
mass	m	kg
time	t	s
distance / displacement	s, x, h, r, etc.	m
speed / velocity	u, v	m s^{-1}
acceleration	a, g	m s^{-2}
force	F, R, W, T, etc.	N
momentum	p	kg m s^{-1} or N s
energy	E	J
work	W	J
power	P	W
impulse	I	kg m s^{-1} or N s
angle	Θ	radians
angular velocity	ω	rad s^{-1}

Compound Units: when the formula being analysed involves a compound unit such as the newton (N) or the joule (J), it is important that these be broken into the basic units.

$$1 \text{ N} = 1 \text{ kg m s}^{-2} \text{ (from } F = ma)$$
$$1 \text{ J} = 1 \text{ kg m}^2 \text{ s}^{-2} \text{ (from } W = Fs)$$
$$1 \text{ W} = 1 \text{ kg m}^2 \text{ s}^{-3} \text{ (from } P = W/t)$$

Circular Motion (radians): The unit of angle, the radian, is a ratio, and therefore does not have a dimension. In a formula with angle involved, or angular velocity, while it is usual to write the unit as the radian (or rad s^{-1} for angular velocity), in dimensional analysis the radian is left out as it is not a unit.

Coefficients of Friction and Restitution: Like radians, the coefficients of friction (μ) and restitution (e) are also ratios, and therefore have no unit, and in dimensional analysis can be ignored.

Trigonometric Ratios and Logarithms: If any formula being used in dimensional analysis involves any of the trigonometric ratios (sin, cos or tan), logarithms (log) or natural logarithms (ln), then these too are ratios and are dimensionless.

Example A1A

Use dimensional analysis to verify the following equation: $s = ut + \frac{1}{2}at^2$.

Solution:

$$s = ut + \tfrac{1}{2}at^2$$

$$\text{m} = \frac{\text{m}}{\text{s}}\,\text{s} + \frac{\text{m}}{\text{s}^2}\,\text{s}^2 \quad \text{(the } \tfrac{1}{2} \text{ is ignored as it is just a coefficient)}$$

$$\text{m} = \text{m} + \text{m} \quad \text{All terms have the same unit}$$

Example A1B

Use dimensional analysis to verify the following equation: $a = r\omega^2$.

Solution: $a = r\omega^2$

$\frac{\text{m}}{\text{s}^2} = \text{m}\left(\frac{\text{rad}}{\text{s}}\right)^2$ radian can be ignored as it is a ratio

$\frac{\text{m}}{\text{s}^2} = \frac{\text{m}}{\text{s}^2}$ All terms have the same unit.

Example A1C

The following formula is used to find velocity of a particle moving with simple harmonic motion: $v = \omega\sqrt{A^2 - x^2}$, where v is the velocity, ω is an angular velocity, A is the maximum amplitude (a length) and x is it's displacement. Use dimensional analysis to show that $\omega\sqrt{A^2 - x^2}$ has the same unit as velocity.

Solution: $v = \omega\sqrt{A^2 - x^2}$

$\frac{\text{m}}{\text{s}} = \frac{\text{rad}}{\text{s}}\sqrt{\text{m}^2 - \text{m}^2}$ radian can be ignored as it is a ratio

$\frac{\text{m}}{\text{s}} = \frac{\text{m}}{\text{s}}$ All terms have the same unit.

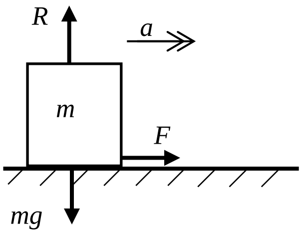
Example A1D

A particle of mass m is at rest on a horizontal turntable at a radius 2 m from the centre of the turntable. The coefficient of friction between the particle and the turntable is μ. The turntable starts rotating horizontally with angular velocity ω. The angular velocity is gradually increased until the particle is on the point of slipping.

(a) Show that the maximum angular velocity achieved, when the particle is on the point of slipping is: $\omega_{\max} = \sqrt{\frac{\mu g}{2}}$.

(b) Use dimensional analysis to show that $\sqrt{\frac{\mu g}{2}}$ has the same unit as angular velocity.

Solution:



(a) Resolve $\uparrow$: $R = mg$

$F = \mu R: \quad \Rightarrow F = \mu mg$

$F = ma \rightarrow: \ \mu mg = ma = mr\omega^2$

$\Rightarrow \omega^2 = \frac{\mu g}{r} = \frac{\mu g}{2}, \qquad \Rightarrow \omega = \sqrt{\frac{\mu g}{2}}$ Q.E.D.

(b) $\omega = \sqrt{\frac{\mu g}{2}}$

$\frac{\text{rad}}{\text{s}} = \sqrt{\frac{\text{m s}^{-2}}{\text{m}}}$ Radian can be ignored as it is a ratio. Also note that in this example the 2 is not a coefficient, but represents the radius and therefore has a unit.

$\frac{1}{\text{s}} = \frac{1}{\text{s}}$ All terms have the same unit.

Exercise A1 – dimensional analysis

1) Use dimensional analysis to show that the units are consistent for the following formulae:

(i) $v = u + at$ (ii) $v^2 = u^2 + 2as$ (iii) $s = ut + \frac{1}{2}at^2$

(iv) $s = \left(\frac{u+v}{2}\right)t$ (v) $s = \left(\frac{t_2+t}{2}\right)v$ (vi) $t_2 = \sqrt{\frac{at^2 - 4s}{a}}$

(vii) $h = \frac{4u^2 - g^2t^2}{8g}$ (viii) $F = ma$ (ix) $F = \mu R$

(x) $h = \frac{u^2 \sin^2\alpha}{2g}$ (xi) $t = \frac{2u\sin\alpha}{g}$ (xii) $s = \frac{u^2 \sin 2\alpha}{g}$

(xiii) $E = mgh$ (xiv) $E = \frac{1}{2}mv^2$ (xv) $W = Fs$

(xvi) $P = \frac{W}{t}$ (xvii) $P = Fv$ (xviii) $p = mv$

(xix) $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ (xx) $I = mv - mu$ (xxi) $-e = \frac{v}{u}$

(xxii) $l = r\theta$ (xxiii) $v = r\omega$ (xxiv) $a = r\omega^2$

(xxv) $a = \frac{v^2}{r}$ (xxvi) $A = \pi r^2$ (xxvii) $V = \frac{4}{3}\pi r^3$

(xxviii) $t = \frac{2\pi r}{v}$ (xxix) $F = k(l - l_0)$ (xxx) $W = \frac{1}{2}kx^2$

(xxxi) $\frac{ds}{dt} = u + at$ (xxxii) $\frac{dv}{dt} = a$ (xxxiii) $v\frac{dv}{ds} = a$

2) A car starts from rest and travels in a straight line on a horizontal road, first with acceleration *a*, and then with deceleration *f*. The car comes to rest when it has travelled a total distance *d*.

(i) Show that the overall time for the journey, *t*, is given by: $t = \sqrt{2d\left(\frac{1}{a} + \frac{1}{f}\right)}$.

(ii) Use dimensional analysis to show that the units are consistent for this expression.

3) In Einstein's famous equation for mass-energy equivalence, $E = mc^2$, *E* represents energy, *m* represents mass and *c* represents the speed of light. Use dimensional analysis to verify that the units are consistent across this equation.

4) A particle of mass *m* is at rest on a horizontal turntable at a radius *r* from the centre of the turntable. The coefficient of friction between the particle and the turntable is μ. The turntable starts rotating horizontally. The angular velocity is gradually increased until the particle is on the point of slipping.

(i) Show that the maximum velocity achieved, when the particle is on the point of slipping is: $v_{max} = \sqrt{\mu r g}$.

(ii) Use dimensional analysis to show that the units for this expression are consistent.

5) A particle of mass *m* is suspended by a light inextensible string from a point P. The particle performs a horizontal circle of radius 1 m, with the string inclined at an angle θ to the vertical.

(i) Show that the angular velocity of the particle, ω, is given by: $\omega = \sqrt{g\tan\theta}$.

(ii) Use dimensional analysis to show that the units for this expression are consistent. (Careful!)

6) The following formula is written in terms of mass *m*, momentum *p*, velocity *v*, displacement *s* and time *t*: $v = \sqrt{\frac{2ps}{mt}}$. Use dimensional analysis to verify that the units are consistent for this formula.

Appendix 2
Guide to the Mathematical Modelling Project

Mathematical Modelling

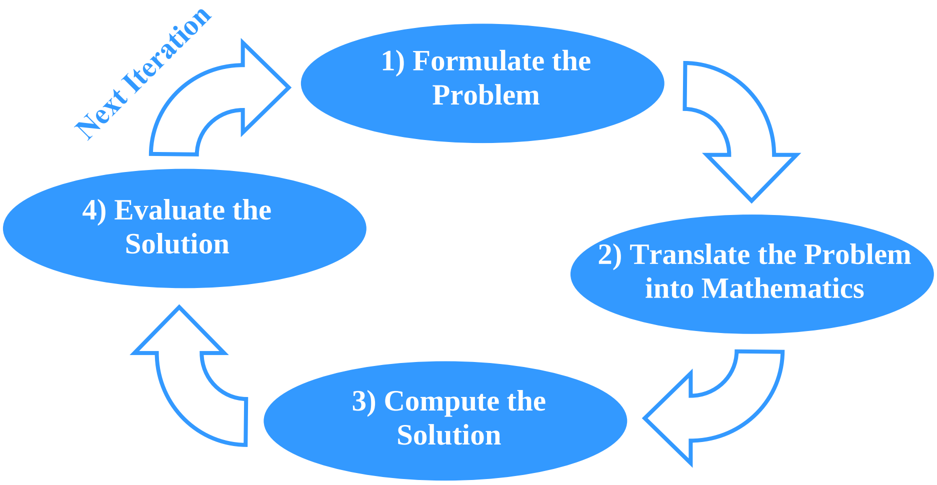
What is Mathematical Modelling?

Mathematical modelling is the process of describing a real world problem in mathematical terms, usually in the form of equations, and then using these equations both to help understand the original problem, and also to discover new features about the problem.

A **mathematical model** is an abstract description of a concrete real world system using mathematical concepts and language. The process of developing a mathematical model is termed mathematical modelling. Mathematical models are used in a very wide, and increasing, number of applications, such as in the natural sciences (physics, biology, earth science, chemistry, medicine) and engineering disciplines (computer science, electrical engineering), as well as in non-physical systems such as the social sciences (economics, psychology, sociology, political science). A model may help to explain a system and to study the effects of different components, and to make predictions about behaviour.

Mathematical models can take many forms, including dynamical systems, statistical models or differential equations. These and other types of models can overlap, with a given model involving a variety of abstract structures. In many cases, the quality of a scientific field depends on how well the mathematical models developed on the theoretical side agree with results of repeatable experiments. Lack of agreement between theoretical mathematical models and experimental measurements often leads to important advances as better theories are developed to explain what is actually happening in the real world.

The Modelling Cycle



There are four key stages to the modelling cycle:
1) Formulate the Problem
2) Translate the Problem into Mathematics
3) Compute the Solution
4) Evaluate the Solution

These four stages are then repeated for each iteration the model goes through. At each iteration the model becomes more complicated, and extra factors are incorporated into the model to hopefully bring the model closer to modelling the real situation.

Formulating the Problem

Formulating the problem means stating very clearly in words what the problem to be modelled is. A key part of this is to list and explain what **assumptions** are being made. Any model always requires assumptions to be made. To formulate any problem, an extensive amount of **research** will be required.

In effect, **assumptions** are aspects of the true model (real world situation) that we are choosing not to incorporate into the current iteration of the model. This is to make the real world situation simple enough for us to model. If we are formulating the problem for the 1st iteration of a model, then the list of assumptions will probably be quite long and is basically a list of all the possible factors which may affect the situation we are modelling. Some of these factors may have a significant effect on the outcome, others may only have a very minor effect. Nonetheless, it is very important that they are all listed. For any subsequent iterations, this usually, though not always, involves incorporating one of the assumptions into the model. Each iteration of a model makes the model more complicated as more and more factors are incorporated into the model.

The key aspects of **researching** a mathematical model are:

- What are the key factors (variables) that are required for the initial iteration of the model? These must be clearly defined. It is vital that the initial iteration is kept as simple as possible. Any complexities can then be added in subsequent iterations.
- What formulae or equations link these factors (variables)? This can be done in the iteration it is being used to avoid repetition.
- What other factors (assumptions) affect the real situation? It is important to do a rough ranking of the significance of these assumptions for the model. This ranking will be used to determine what factors are modelled in subsequent iterations, and in what order.
- What are the specific details (context) required for the model? This is mainly a list of key data required for the situation being modelled. The more specific the model is the better. It is vital that there is no vagueness in the formulation of the problem. If there is any vagueness, the reality is that we really don't know what we are modelling, and therefore can't model it!
- Make sure the research section of your report is precise, concise, and strictly related to the model. There is no need to use up precious word count with niceties!
- It is crucial that all sources of data and formulae are fully recorded and documented. There is more detail on this in a later section.

Throughout this section I will illustrate the main ideas using two different examples. These initial formulations are vague and will be refined before proceeding to the next stage.

- Example 1: throwing a javelin,
- Example 2: a bus route.

Example 1: We must refine the formulation of the problem and add detail to the statement "throwing a javelin." Fundamentally this is a mechanics based situation, where we will primarily be using kinematics and dynamics in our model. We must answer questions such as:

- Who is throwing the javelin?
- Where is it being thrown?
- At what height above the ground is the javelin released?
- What are the dimensions, mass, etc. of the javelin?
- What key data do we need to set up our initial iteration?
- What assumptions are we making, or put another way, what factors are we ignoring in the initial iteration?
- What are we actually trying to find from our model?

In terms of getting the necessary **data** there are two main sources. We can search (probably online) for existing data, or we can generate the data by performing actual measurements. For this example, we will model the javelin as a simple projectile as the initial iteration, and for such a model the key data we need is the speed of projection, the height of projection and the angle of projection, and from this we can calculate the range or maximum height.

When using existing sources of data, whether that be from book sources or online, it is important to be sure that the source of the data is reliable. This is particularly true when using internet sources – there is an awful lot of misinformation on the internet! If the source is something like an academic paper, or some other serious scientific web site, then it is probably correct. If the source is “the ramblings of Uncle Jimmy” or some such, then it is probably best ignored. If you can find multiple sources which are giving the same or similar data, then it is presumably ok. In our example of throwing a javelin, we will almost certainly use an online source to find the speed at which the athlete releases the javelin.

If we are going to generate the data ourselves, this must be done very carefully to ensure it is as accurate as possible. Note that generating your own data can be a very time-consuming exercise, and in the Leaving Certificate project the data collection part is not a very significant part of the process, so if having to generate your own data can be avoided, I would suggest that is the best way to go. In some types of project, it may be inevitable that we have to generate our own data. In our example, we could seek to generate our own data, but this is relatively complex. How could we find the release speed of a javelin? With modern cameras and phones there are apps and so on that can do this, so it is possible, though difficult, that using an app we could do this from a video of someone throwing a javelin.

In our example we must also find the height of release of the javelin, and also how far behind the foul line this takes place. The length of the javelin throw is measured from the foul line, but the javelin is actually released by the athlete somewhere behind this line, as the athlete must decelerate and stop before the foul line for the throw to be considered legal. For example, if we can find a photograph, or a frame from a video, of an athlete at the moment of release of the javelin sideways on, then these values can be found from the photo. Otherwise, we are dependent on an online source having these values already worked out.

This leads to a very important point in the whole process. The brief for the Leaving Certificate project will in general be very broad and non-specific, and we might have a brilliant idea for a project within that brief, but if we can’t find or generate the core data for that idea relatively quickly and easily, then we should be adapting our project to something where the core data can be found more easily. If the brief in this example was about “throwing a projectile” we might find that looking at the shot putt might be far more straightforward than the javelin, and that there is no good reason to spend the extra time and effort persisting with the javelin when there is, say, plentiful data available for the shot putt.

What do we end up with as the formulation of our problem? In this case we may end up with something like: “To find what angle Barbora Spotakova should have thrown her javelin to maximise her world record in Stuttgart in 2008.” This is now our project title. It is crucial that this is very specific. With this we would need the core data, such as her maximum release speed, her release height, and the distance behind the foul line at release. Presuming we can find this data, we would then go on and include the data specifics for her javelin, dimensions, mass, etc. Finally, we would then list off the assumptions we are making for the initial iteration. These would include (this list is not exhaustive):

- Assuming the javelin is a point mass.
- Taking the value of g to be 9.81 m s^{-2}.
- Taking a kinematic approach to the javelin in flight. This means that we are assuming the only force acting on the javelin while it is in flight is its weight. Including the other forces, such as drag or lift, could then be the basis of further iterations.
- Assuming there is no wind.
- Assuming optimal conditions for air density, pressure, humidity, temperature, etc.

Example 2: We must refine the formulation of the problem and add detail to the statement “a bus route.” Fundamentally this is a statistically based situation, though we may bring in some mechanics for the journey times of the bus between stops. We must answer questions such as:

- What type of bus route: school, urban, interurban, etc.?
- How many stops are there on the route?
- What are the distances and journey times between the stops?
- How many passengers are expected to get on/off at each stop, and how long does this take?
- What factors would affect the journey time of the bus?
- What is the maximum passenger capacity of the bus?
- What is the maximum acceleration / deceleration / speed of the bus?
- How do we deal with extreme events such as adverse weather or traffic accidents?

For this type of project, the data will almost certainly have to be self-generated, as we are unlikely to find any reliable data from any other source. It is therefore important to be very realistic about how we define the parameters of our project. Keeping it as local as possible will help, so doing something like a school bus route is probably far more feasible than anything else. Also limiting the number of stops is probably a good idea – extra stops don't add any complexity to the project, but they do add time and effort. This sort of project could also be based on an actual bus route or could be trying to work out the journey time for a proposed bus route that does not yet exist.

The data for a project like this will be largely statistical in nature. For example, to model the number of passengers wishing to get on the bus at a particular stop, we need data like the minimum number, maximum number, and we can then decide which form of average (mean, median or mode) is most appropriate to use. We need to answer questions such as:

- How long does it take one passenger to alight?
- Does it take twice as long for two passengers to alight as one?
- Does it take longer for a passenger with mobility issues to get on the bus? What proportion of passengers have mobility issues?

To model the journey times between stops, this could be done statistically, or using mechanics such as velocity/ time graphs, or even some combination of the two. Whatever decisions you make, the important thing is that they are explained clearly, succinctly, and rationally.

The final formulation of a project of this type might be something like: "To find the latest time the school bus for Route X should arrive at the first stop so that it arrives at school at least 5 minutes before the start of school on at least 98% of days." This would need to be accompanied by detailed data for route X covering stops, passengers, and journey times. For a first iteration, maybe modal times might be used, and then for subsequent iterations maybe more detailed modelling of each of the stages might be done. The key is that all decisions are justified, and all assumptions explained.

Computing the Solution

Once you have formulated the problem thoroughly you should then be able to calculate the solution from the data you have. Depending on the context of the problem this could involve a very wide range of mathematical disciplines, including solving many types of equations (linear, quadratic, simultaneous, etc.), differential equations, difference equations, graph theory methodology or statistical calculations.

In general, the calculations for the first iteration should be relatively straightforward, and at Leaving Certificate level will be worked out by hand on paper. However, for subsequent iterations the calculations can get complicated very quickly and may require the use of online calculating tools or spreadsheets or something similar. There is more on this in later sections.

Example 1: For the first iteration of this example the calculation is just a kinematic projectile calculation to find the angle which gives the largest range and is only a slight variation on calculations that are done in the projectiles section of the course. However, for subsequent iterations the calculations will probably become much more complicated, depending on what changes are made to the model at each iteration. If drag or lift forces are included in the model, this will require use of differential equations, as these are both variable forces. After a couple of iterations, most mathematical models based on mechanics will probably be very difficult to solve manually and will probably require the help of some form of technology.

Example 2: The mathematics involved in a statistical type of model is usually a bit more straightforward than for other types of models. For the initial iteration you will be probably trying to find a modal value for the journey time, and then in subsequent iterations including the effects of various sorts of potential delays to this to find the maximum journey time for 98% of the days. The calculations for this style of model usually don't get too complicated, but something like a spreadsheet could still be very useful. Later iterations may involve the introduction of sub-models, which might, say, just explore how long the bus needs to be stopped for various numbers of passengers to embark, so that the ultimate model ends up being the amalgamation of a series of sub-models.

Evaluating the Solution

As we complete the calculations for each iteration, we must then analyse and evaluate the answer (or answers) we are ending up with. If we are modelling a real-life situation, we will have actual values to compare to. In many cases the values we get from an initial or other early iterations may be substantially different from the real-life values. This is generally because real-life is generally complex, and many factors (iterations) will need to be incorporated into the model for it to give realistic results. In other simpler modelling cases the initial iteration may be giving us a reasonably realistic value. This will vary depending on the nature and complexity of what is being modelled.

The key purpose of the evaluation at the end of each iteration is to see how close we are getting to a useful and accurate model, and if we are not there yet to determine what factor or factors should be incorporated into the next iteration or iterations to get closer to a realistic model. From this the next iteration is formulated. **It is important that a full evaluation is done at the end of each iteration**, and that it is a true analysis of the iteration and not just a summary of what was done.

If we are modelling a situation which does not yet exist in real-life the evaluation phase can be tricky as we do not have real-life values to compare with. How many iterations we require to get a useful answer would depend very much on the nature of the project. If we are making a mathematical model for a future probe which is going to be sent to orbit Mars, we need a model that is as accurate as possible and includes any and every factor we can think of. If we are modelling the journey time for a proposed new school bus route a much less precise answer will probably suffice.

In any real-life mathematical model, it is usual for a large number of iterations to be required for the model to give realistic solutions. On ps. 46 and 47 there is a brief description of the history of a mathematical model with which we are all familiar – the weather forecast. If we consider the work done by Lewis Fry Richardson during the First World War as the initial iteration of the model, the models now being used by meteorologists around the world, which include myriads of sub-models, have probably gone through hundreds of iterations since then and are now so complex that they require some of the most advanced computers in the world to come up with solutions in any sort of reasonable timeframe.

Iterations

In any real modelling exercise, it is usual that many iterations are required to get meaningful results. For the Leaving Certificate it is important to note that the objective is not to necessarily create a model that gives meaningful results, but rather to show a clear understanding and engagement with the process of creating a mathematical model. For the Leaving Certificate project, usually there is no need to go beyond three or four iterations even though at this stage the model may be far from "finished." Make sure to justify and explain the rationale for incorporating the assumption concerned into each new iteration, and also make sure each iteration is adding complexity (and more accuracy) to the model. Avoid trivial iterations.

For many types of mathematical model, each new iteration adds enormously to the complexity of the calculations. After a couple of iterations in such situations the calculations will very often need to be solved using technology. In the Leaving Certificate project, it is probable that the candidates should engage with using technology to do calculations to an extent, but there is no need or advantage to go too far with this.

In a mathematical model where the final model is largely a combination of a variety of sub-models, it can be that many of the sub-models are mathematically very similar to each other. If this situation arises in the Leaving Certificate, it is important that the iterations that a candidate chooses to include in their project are using a variety of mathematical skills, rather than just being variations on basically the same theme.

When is a Mathematical Model complete?

The simple answer to this is never! Real-life is exceedingly complex, and there are usually extra factors that can be incorporated into any existing mathematical model, though their effects may be relatively minor. A better way of phrasing the question might be something like: "when is a mathematical model useful?"

If we go back to the weather forecasting example, the models have been producing useful, though not necessarily fully accurate, results for many decades. However, the science involved is so complex that no weather forecasting model will ever be truly complete. Current models are far more accurate, and therefore useful, than those thirty years ago, and in thirty years' time the models will presumably be far better again than the current ones.

For the Leaving Certificate project, given the timeframe involved, there is no need to go beyond three or four iterations for any model. There is no specific instruction on the number of iterations required. The key thing is for the candidate to choose the iterations which will have the largest effect on their model, and also ones which show a variety of mathematical skills so that the chosen iterations are not merely repetitions of previous iterations with only minor variations. The model in the project may still be far from being useful in the real world, but this is fine within the parameters of the Leaving Certificate. Any project in the Leaving Certificate should include in the analysis for the final iteration what factors should be included for subsequent iterations, to show how the model could be further developed and improved.

The Leaving Certificate Project

The project will be done at some stage during 6th year. It counts for 20% of the total mark, with the final examination accounting for the remaining 80%. The timing may vary from year to year, but generally from receiving the project brief to submission is about 3 months. The most important thing to realise is that **the project must be supervised by your teacher**, and they must sign off that the project is all your own work. Therefore, do not go away and do a lot of work on your own, as they will not sign off on it. Always check with your teacher, but by and large most of the project will be done in school under supervision. There is scope to do a certain amount of research at home. This is to ensure that the project is all your own work and has not been completed by someone else. The State Examinations Commission say the following: "candidates may carry out background research relevant to the brief on their own and/or at home, all other parts of the modelling project and report must be completed under the supervision of the class teacher."

The following is what the State Examination Commission say about the modelling cycle.

The modelling project brief is designed to allow you to engage in the full modelling cycle. Your report should show how you selected the problem(s) relevant to the brief to model, translated the problem(s) into mathematics, computed solution(s), analysed your proposed solution(s), and then iterated (repeated) this modelling process, making refinements based on your iterations before drawing conclusions.

A description of the modelling cycle is as follows:

1) Research the background to the brief so as to determine relevant information and data, and to analyse relevant factors and variables.
2) Break up the topic of the brief into manageable parts and select which problem (or problems) relevant to the brief that you intend to model. If necessary, carry out further research to determine and analyse information, data, factors and variables that are specifically relevant to the problem(s) you have selected.
3) Identify and analyse the assumptions which may be necessary to simplify the problem(s) you have selected.
4) Develop an initial mathematical model – translate the information relevant to each problem, together with any assumptions you have made, into a mathematical model.
5) Compute solution(s) to each problem using appropriate mathematical tools. Such tools may include mathematical methods you have met as part of your studies in Applied Mathematics or elsewhere. They may also include computational technologies, including numerical or graphical techniques for generating solutions.
6) Interpret your mathematical solution(s) in the context of the problem you are modelling.
7) Analyse your solution(s). Examine the sensitivity of your solution(s) to changes in assumptions. Compare your solution(s) to your previous solution(s), to solutions found in research, and/or to real-world data. Use this analysis to refine your model.
8) Iterate the modelling process. You should continue this iterative process to refine and improve your model and your solutions before you draw conclusions.
9) Present a report on your work using the digital reporting booklet provided by the State Examinations Commission, available at www.examinations.ie.

It is very easy for a candidate to be initially overambitious in what they will be able to achieve for their modelling project. Your teacher will allocate a limited amount of class time in which to complete the project, and this is largely the same for all candidates around the country. It is vital that the project is completed within this class time, as only limited amounts of background research is permitted outside of this. It is far more important that your project shows a clear engagement with all stages of the modelling cycle, as outlined above, than that it involves any innovative mathematics, and remember that the objective is to demonstrate a clear understanding of, and engagement with, the modelling cycle, rather than to produce a complete useful mathematical model.

Using Online Technologies in the Project

There are myriads of websites that provide a wide range of tools for various forms of mathematical calculations, and most are either free or available for very little cost. There are also many software packages with similar tools which your school computer system may have access to. It is allowed to use such tools in your project, as long as they are properly credited when used. **If it is possible to do a calculation out manually, then this should be done once** to demonstrate that the candidate has a clear understanding of the mathematics involved, but if there are similar repeat calculations then using some form of technology for these calculations is recommended to save time. There is always a balance to be struck between demonstrating knowledge and skills clearly, and the time constraints of the project.

The sort of things that online tools (or software packages) can be used for are (this is not an exhaustive list):

- Solving equations (linear, quadratic, simultaneous, etc.)
- Differentiation
- Integration
- Matrix calculations
- Statistical calculations
- Drawing graphs

If your project needs a tool, there is a good chance that it is there somewhere – go exploring!

Most real mathematical models end up as computer programs. In the Leaving Certificate, this is a mathematical modelling project, not a computer science project. While there is nothing to say computer programming cannot be used, I would strongly caution against it. Your teacher must sign off that it is your own work and may not have the skills to sign off on a computer program. Also, the person assessing your project is not required to have any programming skills. If there is a way to do it without requiring a program, then that is the way to go. I would only consider writing a computer program if: (i) I cannot think of another way of doing it, (ii) I have very good programming skills, and (iii) the required program can be produced very rapidly. Consult closely with your teacher before embarking on any computer programming.

Using Spreadsheets in the Project

Spreadsheets (such as Microsoft Excel, Google Sheets or Apple Numbers) are tools that have many potential uses within the Leaving Certificate modelling project. You may or may not have prior experience with spreadsheets, but they are relatively straightforward to use, and there are many tutorials available to get you up to speed relatively quickly.

All spreadsheets are based on a matrix of cells, and into each cell you can enter text, numbers (data) or formulae. Each cell has a unique reference – cell D12 is in column D and row 12. It is by using formulae that the real power of spreadsheets can be unlocked. There may be slight differences in the syntax for the formulae between the different spreadsheet packages, but for the most part they are the same. Any formulae given below are for Microsoft Excel, but should be either the same, or very similar for the other packages. What is below is only a brief introduction to the use of spreadsheets to give some suggestions as to how they might be used as part of the mathematical modelling project, but really the possibilities are endless. Go and explore for yourselves.

Any spreadsheet formula starts with an =. The symbols for the four basic arithmetical operations are: plus (+), minus (-), multiply (*) and divide (/). For example, if in cell C6 we want to put a formula to take the value in cell B6, multiply it by 2 and divide the result by 3, the formula in C6 would be: =(B6*2)/3. The spreadsheet packages also have all the mathematical functions you are likely to need built in. For example, if in cell E5 we want a formula to find the square root of the cube of the value in cell E4 then the formula in E5 is: =SQRT(E4^3). For another example, if in cell F12 we need to find the tan of the angle in cell F11, which is in degrees, then the formula in F12 would be: =TAN(F11*PI()/180). Note that the built in trigonometric functions assume any angles are in radians, so the angle in degrees needs to be converted to radians. It should be easy to find any functions you need with a quick search, and their corresponding syntax.

General tips for using formulae in spreadsheets:

- Be very careful with the syntax of your formulae. Everything must be precise, and if it isn't, the formula won't work.
- Brackets are very useful but be careful – for every opening bracket there must be a corresponding closing bracket.
- For a complicated formula it may be simpler to break the calculation down and spread it across several cells, rather than trying to do it all in one cell.
- Once a formula is correctly structured, it can be copied any number of times to other places where the same calculation is required.
- Spreadsheets, like calculators, give answers to a large number of significant figures. If the core data you are using is only valid to, say, three significant figures, it is wrong to give any results to any more than three significant figures. Think in terms of significant figures, rather than decimal places.

Another potential use of spreadsheets is to use a "trial and error" method to solve equations which are otherwise unsolvable by conventional methods or require methods beyond the scope of Leaving Certificate mathematics. In later iterations of mathematical modelling such situations are common. To illustrate how this might be done we reconsider the example of throwing a javelin.

If we assume the javelin is being thrown from a height of 2 m with a speed of 30 m s^{-1}, and we are trying to find at what angle it should be thrown to maximise the range. For an initial iteration based on just kinematics it was found that the optimum angle was 44.4° to give a range of 93.8 m. For the second iteration we are adding a drag force in the horizontal direction only. For this we are just using standard kinematics in the vertical direction to find the time of flight. This gives: $t = \dfrac{30\sin\alpha + \sqrt{(30\sin\alpha)^2 + 4g}}{g}$. Without going into details of how it was derived, when including drag in the horizontal direction, we get a formula for range of: $R = 30\ln(1 + t\cos\alpha)$. To derive this formula requires the solving of a differential equation as drag is a variable force which varies with velocity.

The objective here is to find which value of α gives maximum range, and what is that maximum range. In the spreadsheet shown to the right this has been calculated for a range of angles. In the previous iteration the optimum angle was 44.4°, so here we have started with 43.5° as we assume it won't change much in this iteration. The formula in cell A3 is: =A2+0.1. This is then filled down for the rest of column A to give angle increments of 0.1°. The other formulae used are as follows:

- Cell B2: =(A2*PI())/180
- Cell C2: =30*SIN(B2)
- Cell D2: =SQRT((C2^2)+(4*9.81))
- Cell E2: =(C2+D2)/9.81
- Cell F2: =30*LN(1+(E2*COS(B2)))

	A	B	C	D	E	F
1	Angle	Radians	30sina	Surd	Time	Range
2	43.5	0.759218	20.65064	21.57982	4.304838	42.49466
3	43.6	0.760964	20.68859	21.61614	4.312409	42.49689
4	43.7	0.762709	20.72647	21.65241	4.319967	42.49885
5	43.8	0.764454	20.7643	21.68861	4.327514	42.50054
6	43.9	0.7662	20.80205	21.72477	4.335048	42.50197
7	44	0.767945	20.83975	21.76086	4.34257	42.50313
8	44.1	0.76969	20.87738	21.79691	4.350081	42.50403
9	44.2	0.771436	20.91495	21.83289	4.357579	42.50467
10	44.3	0.773181	20.95246	21.86883	4.365065	42.50503
11	44.4	0.774926	20.9899	21.9047	4.372538	42.50514
12	44.5	0.776672	21.02728	21.94052	4.38	42.50497
13	44.6	0.778417	21.06459	21.97628	4.387449	42.50455
14	44.7	0.780162	21.10184	22.01199	4.394886	42.50385
15	44.8	0.781908	21.13903	22.04764	4.402311	42.50289
16	44.9	0.783653	21.17615	22.08323	4.409723	42.50167
17	45	0.785398	21.2132	22.11877	4.417123	42.50018

From the spreadsheet the optimum angle is still 44.4°, but the range has reduced to 42.5 m. This change in the range seems too extreme for something as aerodynamic as a javelin, so as part of the evaluation of this iteration it would be crucial to assess whether the drag force is being calculated accurately. This example is just to illustrate how a spreadsheet might be used in a trial and error type calculation, and the numbers used are somewhat arbitrary. To maximise the range equation in relation to the angle using just calculus is probably well beyond the mathematical abilities of a typical Leaving Certificate candidate.

Obviously, spreadsheets are of use for the calculation section of the various iterations, but they may also be of use in the final reporting stage. Various sorts of graph (called charts in spreadsheet packages) can be created easily using spreadsheets, and if any such graphs are useful to demonstrate clearly the results of the mathematical model, then spreadsheets may be the easiest way to create these graphs.

Writing up the Project Report

The report is the most important part of your modelling project. This is crucial for candidates to remember. It is very easy while doing the project to get very hung up on the iterations and calculations and so on, but it is the report that is submitted and assessed, and candidates must give it the time and effort it deserves. Do not leave it to the last few days before the deadline.

The report is submitted electronically using a document created in Microsoft Word. The State Examinations Commission will provide a template booklet which must be used. This booklet is largely the same from year to year, so if you go to www.examinations.ie you should be able to find a booklet from a previous year to have a good look at. When your report is complete it must be saved in pdf form, and there will be very specific instructions from the State Examinations Commission as to what your file should be named. Make sure to comply with these instructions fully.

Microsoft Word is a very powerful tool. This whole textbook was created using Microsoft Word (except the cover). As well as basic text, your report will inevitably include images of various sorts, as well as equations. There is a tool within Word called the Equation Editor, and this should be used to type any equations and formulae into your report. Note that typing equations is a slow process, and for something like a fully worked solution it may be better to handwrite the solution clearly, scan it and include it as an image rather than typing it. It is always a matter of balance, and what takes less time. You will be under time pressure to get the project done.

The first number of pages of the booklet are largely instructions. Make sure to type in your examination number and date of birth on the first page and read all the instructions carefully. In the following paragraphs we will look in detail at the most important of those instructions.

1) Your report must not exceed 900 words (excluding references, equations, diagrams, graphs, etc.).

   This is probably the most crucial, and difficult, instruction to comply with. 900 words is very few. For example, there are pages within this textbook with over 600 words on a single page. In writing your report you are aiming for a style of language that is very **precise** and **concise**. There is literally no room for any excess words. The first draft of your report is very likely to exceed the permitted word count, and you will need to edit carefully to remove unnecessary words and rephrase sentences so that they explain what is required with the minimum number of words.

2) Your report must not include more than 20 images. An image can be any relevant table, graph, chart, diagram or photograph.

   In general, this is not problematic. It is unlikely that you would need more than 20 images. Images can be of a very wide variety. As well as things like photographs, Excel graphs, tables, etc., you could be including such things as screenshots from Excel, screenshots from various online maths tools, scans of handwritten maths or scans of hand-drawn diagrams. It is unlikely that you will need more than 20. Make sure to limit any text within images to just what is necessary. Do not try to circumvent the word count by putting extra text in images.

3) When referring in the body of the report to any specific image, the image must be properly labelled (Figure 1, Figure 2, etc.). Images should not be used as a means to include additional text. It is advisable not to use images where a person or persons in these images may be identifiable.

Just make sure all images are properly labelled.

4) The total file size of your digital report including all embedded images must not exceed 100 MB. Videos must not be included in your report.

This is generally not a problem to comply with, but you must check. If your file size is too big, you will need to reduce the resolution of your images. Images take much more file size as compared to text or equations.

5) You must not change the structure or format of the booklet in any way and should adhere to the following formatting guidelines:

- The text should be in Arial, font size 12.
- You may make use of text editing features such as italics, bullets, etc.
- Document margins have been set and should not be changed.

This instruction is reasonably straightforward, and just needs to be complied with. Note that some versions of the Equation Editor do not allow you to change the font, but this is ok within equations. Using bulleted or numbered lists is often a very useful way to keep the word count down.

6) You must reference any information used in your report, such as: publications – including books, professional journals and government reports; online sources and other types of media; specialist organisations and relevant individuals. To include such material without properly referencing the source will be considered plagiarism.

This is crucial. From the very start, make a note of any sources you use. It is particularly important to reference any online sources correctly. Try to give as complete a web address for any online sources as possible. Note that any references are not included in the word count. For any source from a book or academic journal, give the book or journal name, article name if appropriate, author(s), and page number if possible. In general, make it as easy as possible for someone to find the same information you have used quickly. Your references can be included into your report in a number of ways: (i) Microsoft Word includes tools to create endnotes (at the end of the document) or footnotes (at the bottom of the current page), (ii) a list of references at the end of each section of your report, (iii) in brackets at the point in the report where the source is used.

After the section of the report with the instructions, there is a page for the project title. As discussed before, this is basically the formulation for your initial iteration, including references to the factors included in your later iterations. This title should be detailed and specific, not vague and woolly! Note that this page is not included in your overall word count. Following the title page is a page on which you must record your word count for the following sections of the report. It is important that these word counts are recorded accurately, as they may be checked by the examiners.

The main body of your report then follows in 3 separate sections. The following table shows what should be included in each section, and how the marks are allocated across the sections. You must record the word count for each section separately, and there is no guideline for how many words each section should have, but the total word count must not exceed 900 words. For each section it is vital that all the required content from the table below is included. The first section on *Introduction and Research* and the third section on the *Interpretation of Results* are likely to require more words than the middle section on *The Modelling Process*, as this section will have all the mathematical content, but doesn't usually require so many words. There is some crossover of content allowed between sections. Due to the very restricted word count, it is vital that you do not repeat things that have already been covered. Make sure all the points in the *Indicative Content* column below are covered, but they can be covered in whichever section seems most appropriate.

Section	**Indicative Content**	**Marks**
Introduction and Research	• Background research on brief • Identify specific problem(s) to be modelled • Research specific problem(s) • Identify relevant variables • Present relevant data • Provide citations and references	**20**
The Modelling Process	• Explain and justify model and assumptions • Compute solutions • Present solutions using appropriate mathematical and graphical representations • Analysis of solution(s) – sensitivity to changes in assumptions; comparison with other solutions or real-world data • Iterative process	**50**
Interpretation of Results	• Interpretation of solution(s) in real-world context • Conclusions and reflections	**15**
Communication and Innovation	This is not a distinct section of the report. • Innovative and creative approaches • Overall coherence	**15**

In the booklet template, each of the three sections above has a one page section for you to enter your report. As you fill up the page, the template will automatically create a new page. Most, if not all, of the three sections of your report will require more than a single page, depending on the number of images and so on.

The final requirement listed in the *Indicative Content* column is about Overall Coherence. This is important. It goes back to the title of your model and your initial iteration, and checks that each of your iterations is developing this model further. Your iterations should always be about developing the previous iterations further, and not about modelling a different situation. Make sure from the very start that you formulate your project very clearly, and that all your iterations are developments of this same model.