Chapter 4
Relative Velocity

Section 4A:
The Velocity of One Moving Body Relative to Another

Relative velocity concerns how one moving body appears to move relative to another moving body. Imagine you are in a bus travelling along a straight road at a constant speed of 15 m s^{-1}. A car, travelling in the same direction at a constant speed of 25 m s^{-1}, overtakes the bus. One second after it has overtaken the bus it will be 10 m ahead, after 2 seconds it will be 20 m ahead, and so on. The velocity of the car relative to the bus is 10 m s^{-1}, i.e. every second the car travels ten metres more than the bus travels. To a person sitting in the bus, it looks like the car is moving away from the bus at 10 m s^{-1}. If we say they are both moving in the $\vec{i}$ direction, then the velocity of the bus, $\vec{v}_B$, is $15\vec{i}$ m s^{-1}, and the velocity of the car, $\vec{v}_C$, is $25\vec{i}$ m s^{-1}.

We would write this:

The velocity of the bus: $\vec{v}_B = 15\vec{i}$ m s^{-1}

The velocity of the car: $\vec{v}_C = 25\vec{i}$ m s^{-1}

The velocity of the car relative to the bus, which we call $\vec{v}_{CB} = \vec{v}_C - \vec{v}_B$

$$\therefore \vec{v}_{CB} = 25\vec{i} - 15\vec{i} = 10\vec{i} \text{ m s}^{-1}$$

The general formula for relative velocity is:

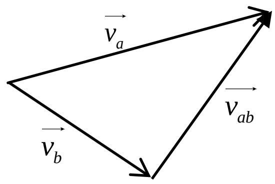
$$\overrightarrow{\boldsymbol{v}_{ab}} = \overrightarrow{\boldsymbol{v}_a} - \overrightarrow{\boldsymbol{v}_b}$$

where: $\overrightarrow{v_a}$ is the velocity of body *a*

$\overrightarrow{v_b}$ is the velocity of body *b*

$\overrightarrow{v_{ab}}$ is the velocity of body *a* relative to body *b*.

We usually show the different velocities as vectors, as shown below:



What this means is that if a person standing on body *b* was observing body *a*, it would look as if body *a* was moving in the direction $\vec{v}_{ab}$.

Solution Strategy for general relative velocity questions:

1) Write the **velocities of both** bodies in terms of $\vec{i}$ **and** $\vec{j}$.
2) Use the relative velocity formula, $\overrightarrow{\boldsymbol{v}_{ab}} = \overrightarrow{\boldsymbol{v}_a} - \overrightarrow{\boldsymbol{v}_b}$, to calculate the velocity of one body relative to the other.

Example 4A1

A car and a lorry are a distance *x* apart on a long straight road, heading towards each other. The car is travelling at a constant speed of 15 m s^{-1} and the lorry is travelling at a constant of speed 12 m s^{-1}.

(i) Calculate the magnitude of the velocity of the lorry relative to the car.

(ii) If the car and lorry pass each other after 40 s, find the value of *x*.

Solution:

(i) Since the car and lorry are moving in opposite directions, their velocities must have opposite signs.

$\overrightarrow{v_C} = 15\vec{i}$ m s^{-1}, $\Rightarrow \overrightarrow{v_L} = -12\vec{i}$ m s^{-1}.

$\overrightarrow{v_{LC}} = \overrightarrow{v_L} - \overrightarrow{v_C} = -12\vec{i} - 15\vec{i}$ m s^{-1} $= -27\vec{i}$ m s^{-1}

$\Rightarrow \left|\overrightarrow{v_{LC}}\right| = 27$ m s^{-1}

(ii) Distance = speed × time: $s = ut = 27(40) = 1080$ m $= 1.08$ km.

Example 4A2

A river is flowing with a velocity of 3 m s^{-1} parallel to the straight banks. The speed of a swimmer in still water is v m s^{-1}. The swimmer swims parallel to the straight banks. On reaching a distance of 120 m swimming against the current, the swimmer immediately turns and swims the same 120 m with the current. The total time for swim is 75 seconds. Find the value of v.

Solution:

The speed in still water, v, is the magnitude of the velocity of the swimmer relative to the river.

When the swimmer is swimming upstream:

$\overrightarrow{v_R} = -3\vec{i}$ m s^{-1}, $\Rightarrow \overrightarrow{v_{SR}} = v\vec{i}$ m s^{-1}.

$\overrightarrow{v_{SR}} = \overrightarrow{v_S} - \overrightarrow{v_R}$, $\Rightarrow \overrightarrow{v_S} = \overrightarrow{v_{SR}} + \overrightarrow{v_R} = (v-3)\vec{i}$ m s^{-1}

When the swimmer is swimming downstream:

$\overrightarrow{v_R} = 3\vec{i}$ m s^{-1}, $\Rightarrow \overrightarrow{v_{SR}} = v\vec{i}$ m s^{-1}.

$\overrightarrow{v_{SR}} = \overrightarrow{v_S} - \overrightarrow{v_R}$, $\Rightarrow \overrightarrow{v_S} = \overrightarrow{v_{SR}} + \overrightarrow{v_R} = (3+v)\vec{i}$ m s^{-1}

$\text{Time} = \dfrac{\text{Distance}}{\text{Speed}}$: $\Rightarrow$ time upstream $t_1 = \dfrac{120}{v-3}$ and time downstream $t_2 = \dfrac{120}{3+v}$.

$t_1 + t_2 = 75$, $\Rightarrow \dfrac{120}{v-3} + \dfrac{120}{3+v} = 75$, $\Rightarrow 120(3+v) + 120(v-3) = 75(v-3)(3+v)$,

$\Rightarrow 8(3+v) + 8(v-3) = 5(v-3)(3+v)$, $\Rightarrow 24 + 8v + 8v - 24 = 5v^2 - 45$,

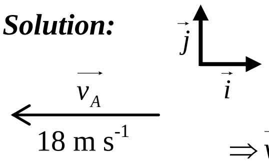
$\Rightarrow 0 = 5v^2 - 16v - 45$, $\Rightarrow (5v+9)(v-5) = 0$, $\Rightarrow v = -1.8$ m s^{-1} or $v = 5$ m s^{-1}

A negative v does not make sense, $\Rightarrow v = 5$ m s^{-1}

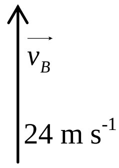
Example 4A3

Ship A is travelling with a speed of 18 m s^{-1} in the direction due West.
Ship B is travelling with a speed of 24 m s^{-1} in the direction due North.
Find the velocity of ship A relative to ship B.

Solution:



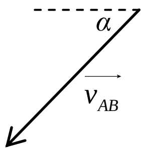
$\Rightarrow \overrightarrow{v_A} = -18\vec{i}$ m s^{-1}



$\Rightarrow \overrightarrow{v_B} = 24\vec{j}$ m s^{-1}

$\overrightarrow{v_{AB}} = \overrightarrow{v_A} - \overrightarrow{v_B}$

$\Rightarrow \overrightarrow{v_{AB}} = -18\vec{i} - 24\vec{j}$ m s^{-1}



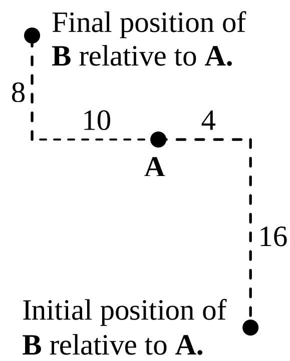
$\left|\overrightarrow{v_{AB}}\right| = \sqrt{18^2 + 24^2} = 30$ m s^{-1}

$\tan\alpha = \dfrac{24}{18} = \dfrac{4}{3}$, $\Rightarrow \alpha = \tan^{-1}\dfrac{4}{3} = 53.1^\circ$ south of west.

Example 4A4
Ship A is travelling due east with a constant speed of 12 km hr^{-1}.
Ship B is travelling with constant velocity.
At midday, the radar screen of ship A shows the position of ship B relative to ship A as $4\vec{i}-16\vec{j}$ kilometres.
Two hours later, the radar shows that the position of ship B relative to ship A is $-10\vec{i}+8\vec{j}$ kilometres.

(i) Write down the velocity of ship A in terms of $\vec{i}$ and $\vec{j}$.
(ii) Show that the change in the position of ship B relative to ship A in the two hours is $-14\vec{i}+24\vec{j}$ kilometres.
(iii) Find the velocity of ship B relative to ship A.
(iv) Find the speed and direction of B, giving the direction to the nearest degree.

Solution:



(i) $\vec{v}_A = 12\vec{i}$ km hr^{-1}.

(ii) The change in the $\vec{i}$ direction from $4\vec{i}$ to $-10\vec{i}$ is $-14\vec{i}$.

The change in the $\vec{j}$ direction from $-16\vec{j}$ to $8\vec{j}$ is $24\vec{j}$.

$\therefore$ the change in position of ship B relative to ship A is $-14\vec{i}+24\vec{j}$ km.

(iii) Since the change in position above happens over 2 hours, the velocity of B relative to A is half of this. $\therefore \vec{v}_{BA} = \frac{1}{2}\left(-14\vec{i}+24\vec{j}\right) = -7\vec{i}+12\vec{j}$ km hr^{-1}.

(iv) $\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$, $\quad \therefore \vec{v}_B = \vec{v}_{BA} + \vec{v}_A$

$\therefore \vec{v}_B = -7\vec{i}+12\vec{j}+12\vec{i} = 5\vec{i}+12\vec{j}$ km hr^{-1}.

$\therefore \left|\vec{v}_B\right| = \sqrt{5^2+12^2} = \sqrt{25+144} = \sqrt{169} = 13$ km hr^{-1}.

and $\tan\alpha = \dfrac{12}{5} = 2.4 \qquad \therefore \alpha = \tan^{-1} 2.4 = 67^\circ$

$\therefore$ ship B is travelling at 13 km hr^{-1}, 67° North of East.

Exercise 4A – general problems involving relative velocity

1) A bicycle is travelling on a straight road at 2 m s^{-1}. A motorbike, travelling on the same straight road in the same direction as the bicycle at 4 m s^{-1}, is 20 m behind the bicycle at a particular instant.
   (i) Find the velocity of the motorbike relative to the bicycle.
   (ii) Find the distance between the motorbike and the bicycle after 3.5 s.
   (iii) Find how long it takes the motorbike to catch up with the bicycle.

2) Two runners are running in a race along a straight road which heads north. At a certain instant, athlete P is *d* metres from the finishing line and is running with a constant speed of 7 m s^{-1}. At this instant athlete Q is 8 metres behind P and is running with a constant speed of 9 m s^{-1}. Q just catches P on the finishing line, and the race ends in a dead heat.
   (i) Find the velocity of Q relative to P.
   (ii) Find the value of *d*.

3) Two lorries, A and B, are travelling along a straight level road in opposite directions. A has a constant speed of 15 m s^{-1} and B has a constant speed of 20 m s^{-1}.
   (i) Find the speed of B relative to A.
   (ii) At a certain instant, the two lorries are 770 m apart. Find how long it takes until the two lorries pass each other.

4) A river is flowing with a velocity of 2 m s^{-1} parallel to the straight banks. The speed of a swimmer in still water is *v* m s^{-1}. The swimmer swims parallel to the straight banks. On reaching a distance of 90 m

swimming against the current, the swimmer immediately turns and swims the same 90 m with the current. The total time for swim is 60 seconds. Find the value of v.

5) Ship A is travelling with a speed of 8 m s^{-1} in the direction due West.
Ship B is travelling with a speed of 15 m s^{-1} in the direction due South.
Find the velocity of ship A relative to ship B, giving the answer in magnitude and direction form.

6) A ship A is travelling north east at 20 km hr^{-1}. Another ship B is travelling at 30 km hr^{-1} in a direction 60° north of west. Find the magnitude and direction of the velocity of A relative to B.

7) Ship P is sailing at 25 km hr^{-1} in the direction $\tan^{-1}\frac{3}{4}$ south of east, and ship Q is sailing at 39 km hr^{-1} in the direction $\tan^{-1}\frac{12}{5}$ north of east. Find the speed and direction of ship Q relative to ship P.

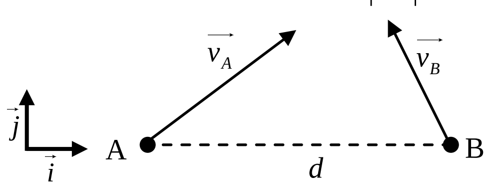
8) The velocity of boat A is $4\vec{i}+8\vec{j}$ km hr^{-1}. The velocity of boat B is $p\vec{i}+q\vec{j}$ km hr^{-1}.
The velocity of A relative to B is $-2\vec{i}+9\vec{j}$ km hr^{-1}. Find the values of p and q.

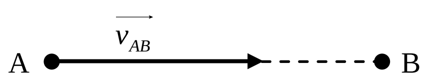
9) The velocity of an aeroplane relative to the wind is due North at 150 km hr^{-1}. If the velocity of the wind is due West at 25 km hr^{-1}, find the velocity of the aeroplane. Give the answer in terms of magnitude and direction.

10) Boat A is travelling due north with a constant speed of 10 km hr^{-1}. Ship B is travelling with constant velocity. At 10:00, the radar screen of ship A shows the position of ship B relative to ship A as $-5\vec{i}+2\vec{j}$ kilometres. Three hours later, the radar shows that the position of ship B relative to ship A is $13\vec{i}-4\vec{j}$ kilometres.
    (i) Show that the change in the position of ship B relative to ship A in the three hours is $18\vec{i}-6\vec{j}$ kilometres.
    (ii) Find the velocity of ship B relative to ship A in terms of $\vec{i}$ and $\vec{j}$.
    (iii) Find the speed and direction of B, giving the direction to the nearest degree.

Section 4B: Collisions and Interceptions

One type of problem we must deal with in this topic is to find out in what direction one body should travel so that it collides or intercepts with another moving body, which started from a different place. In order to solve these problems, either the two bodies will start east and west of each other, or north and south of each other. If they are east and west of each other, we make this the $\vec{i}$ direction. For the two bodies to intercept each other, their velocities in the direction perpendicular to the line joining them (the $\vec{j}$ direction) must be equal. If they are north and south of each other, we make this the $\vec{j}$ direction. For the two bodies to intercept each other, their velocities in the direction perpendicular to the line joining them (the $\vec{i}$ direction) must be equal. In the example below, two bodies, A and B, are a certain distance d apart, and are travelling with velocities $\overrightarrow{v_A}$ and $\overrightarrow{v_B}$ respectively. For these two bodies to intercept, or collide, the components of their velocities in the $\vec{j}$ direction must be equal. If this is so, then the velocity of A relative to B, $\overrightarrow{v_{AB}}$, will only have an $\vec{i}$ component. This means that $\overrightarrow{v_{AB}}$ will be directed along the line joining A and B, pointing towards B. To find when the collision will occur, just divide the original distance apart, d, by the magnitude of the relative velocity, i.e. $t=\dfrac{d}{\left|\overrightarrow{v_{AB}}\right|}$





Solution Strategy for problems with interception:

1) Draw a **detailed vector diagram** of the problem.
2) For **interception, the components of the two velocities of the two objects in the direction perpendicular to the line joining the two objects must be equal**. Put the $\vec{i}$ or $\vec{j}$ axis along the line joining the two objects.
3) The **time to collision** is calculated by dividing the original distance apart by the magnitude of the relative velocity.

Example 4B1

Two planes A and B are flying on straight courses at the same height above ground level. At a certain time A is 160 km due west of B. Plane A is travelling at a speed of 400 km hr^{-1} at an angle θ° north of east, where $\tan\theta = \frac{3}{4}$. Plane B is travelling due north at 240 km hr^{-1}.

(i) Draw a diagram showing the relative positions of A and B.

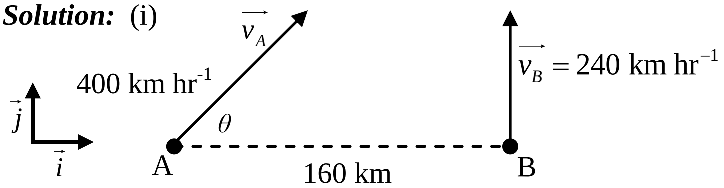
(ii) Find the velocity of A in terms of $\vec{i}$ and $\vec{j}$.

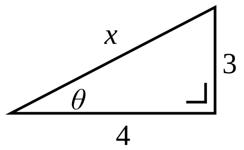
(iii) Find the velocity of B in terms of $\vec{i}$ and $\vec{j}$.

(iv) Find the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$.

(v) How long does it take A to intercept B?

Solution: (i)





Pythagoras: $x^2 = 3^2 + 4^2$, $\Rightarrow x = 5$

$\Rightarrow \sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$

(ii) $\vec{v_A} = 400\cos\theta\vec{i} + 400\sin\theta\vec{j} = 400\left(\frac{4}{5}\right)\vec{i} + 400\left(\frac{3}{5}\right)\vec{j} = 320\vec{i} + 240\vec{j}$ km hr^{-1}

(iii) $\vec{v_B} = 240\vec{j}$ km hr^{-1}

(iv) $\vec{v_{AB}} = \vec{v_A} - \vec{v_B} = \left(320\vec{i} + 240\vec{j}\right) - 240\vec{j} = 320\vec{i}$ km hr^{-1}

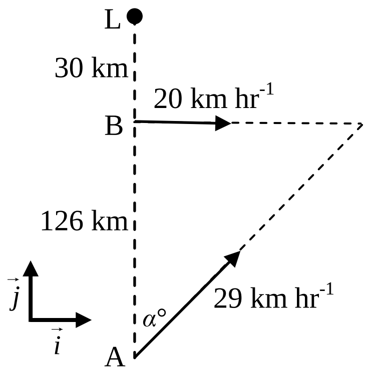
$\Rightarrow \left|\vec{v_{AB}}\right| = 320$ km hr^{-1}

(v) $t = \dfrac{d}{\left|\vec{v_{AB}}\right|} = \dfrac{160 \text{ km}}{320 \text{ km hr}^{-1}} = 0.5 \text{ hr} = 30 \text{ mins.}$

Note that $\vec{v_{AB}}$ is in the $\vec{i}$ direction only. For all interception questions the relative velocity is always along the line joining the two particles.

Example 4B2

Ship A is positioned 126 km due south of ship B, and ship B is positioned 30 km due south of lighthouse L, as shown in the diagram. Ship A is moving at an angle α° east of north with a constant speed of 29 km hr^{-1}. Ship B is moving due east at a constant speed of 20 km hr^{-1}.



Find: **(i)** the velocity of B in terms of $\vec{i}$ and $\vec{j}$,

(ii) the velocity of A in terms of α, $\vec{i}$ and $\vec{j}$,

(iii) the size of angle α, to the nearest 0.1°,

(iv) the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$.

Ship A intercepts ship B after *t* hours.

Find: **(v)** the value of *t*,

(vi) the distance from lighthouse L to the meeting point, to one place of decimals.

Solution: (i) $\vec{v_B} = 20\vec{i}$ km hr^{-1}, (ii) $\vec{v_A} = 29\sin\alpha\vec{i} + 29\cos\alpha\vec{j}$ km hr^{-1}

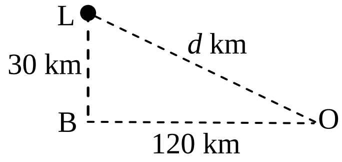
(iii) For interception: $\vec{i}$ components equal: $\Rightarrow 29\sin\alpha = 20$

$\Rightarrow \alpha = \sin^{-1}\dfrac{20}{29} = 43.6°$

(iv) $\overrightarrow{v_{AB}} = \overrightarrow{v_A} - \overrightarrow{v_B} = \left(29\sin 43.6^\circ \vec{i} + 29\cos 43.6^\circ \vec{j}\right) - 20\vec{i} = 21\vec{j}$ km hr^{-1}

$\Rightarrow \left|\overrightarrow{v_{AB}}\right| = 21$ km hr^{-1}

(v) $t = \dfrac{d}{\left|\overrightarrow{v_{AB}}\right|} = \dfrac{126 \text{ km}}{21 \text{ km hr}^{-1}} = 6$ hrs.



(vi) In 6 hours, B travels: $s = ut = 20(6) = 120$ km

$\Rightarrow$ Interception occurs at O where $|\text{BO}| = 120$ km

Pythagoras: $d^2 = 120^2 + 30^2 = 15300$

$\Rightarrow$ distance from lighthouse to meeting point $= |\text{LO}| = d = 123.7$ km

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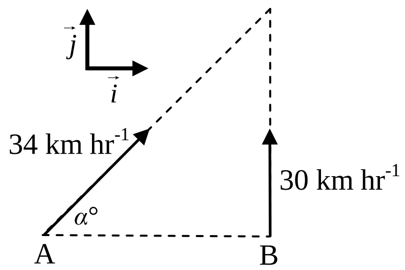
Exercise 4B – relative velocity problems involving interceptions and collisions

1) Ship A is 112 km due west of ship B. A is moving at a constant speed of 34 km hr^{-1} in the direction east α° north where $\tan\alpha = \frac{15}{8}$. B is moving due north at a constant speed of 30 km hr^{-1}.

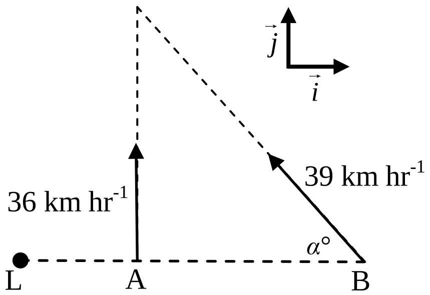


Find: (i) the velocity of A in terms of $\vec{i}$ and $\vec{j}$,
(ii) the velocity of B in terms of $\vec{i}$ and $\vec{j}$,
(iii) the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$.

Ship A intercepts ship B after t hours.

Find: (iv) the value of t,
(v) the distance each ship travels in this time t.

2) Ship A is 90 km due west of ship B. Lighthouse L is 40 km due west of ship A. A is moving due north at a constant speed of 36 km hr^{-1}. B is moving at a constant speed of 39 km hr^{-1} in the direction west α° north where $\tan\alpha = \frac{12}{5}$.

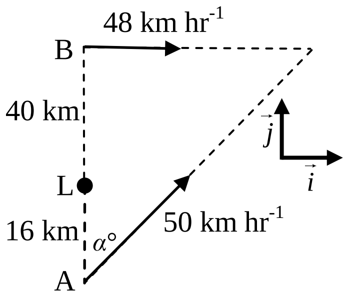


Find: (i) the velocity of A in terms of $\vec{i}$ and $\vec{j}$,
(ii) the velocity of B in terms of $\vec{i}$ and $\vec{j}$,
(iii) the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$.

Ship B intercepts ship A after t hours.

Find: (iv) the value of t,
(v) the distance from lighthouse L to the meeting point, to the nearest kilometre.

3) Ship A is positioned 16 km due south of lighthouse L. A is moving at a constant speed of 50 km hr^{-1} in the direction α° east of north where $\tan\alpha = \frac{24}{7}$. Ship B is positioned 40 km due north of lighthouse L, and is moving due east at a constant speed of 48 km hr^{-1}.



Find: (i) the velocity of A in terms of $\vec{i}$ and $\vec{j}$,
(ii) the velocity of B in terms of $\vec{i}$ and $\vec{j}$,
(iii) the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$.

Ship A intercepts ship B after t hours.

Find: (iv) the value of t,
(v) the distance from lighthouse L to the meeting point, to the nearest kilometre.

4) At a particular instant on a rugby pitch, player B is 18 m due north of player A. Player B is running due east at a constant speed of 6 m s^{-1}. Player A is running at a constant speed of 7.5 m s^{-1} in the direction α° east of north, in order to intercept player B.

(i) Draw a diagram showing the initial positions of players A and B.

Find: (ii) the velocity of B in terms of $\vec{i}$ and $\vec{j}$,

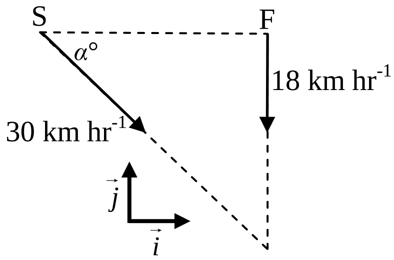
(iii) the velocity of A in terms of α, $\vec{i}$ and $\vec{j}$,

(iv) the value of α, to the nearest 0.1°,

(v) the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$,

(vi) the time taken for A to intercept B.

5) A ferry F is travelling due south with a constant speed of 18 km hr^{-1}. A speedboat S is travelling in the direction α° south of east with a constant speed of 30 km hr^{-1}, and is on a path which will result in it intercepting the ferry F. At a particular instant S is 6 km due west of F.



Find: (i) the velocity of F in terms of $\vec{i}$ and $\vec{j}$,

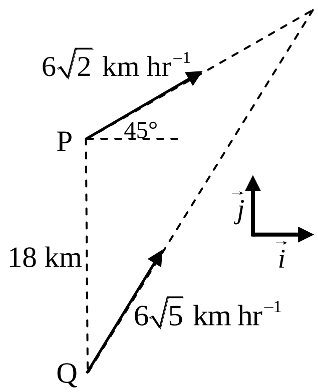
(ii) the velocity of S in terms of α, $\vec{i}$ and $\vec{j}$,

(iii) the value of α, to the nearest 0.1°,

(iv) the velocity of S relative to F in terms of $\vec{i}$ and $\vec{j}$,

(v) the time taken for S to intercept F.

6) At a certain instant ship Q is 18 km due south of ship P. Ship P is travelling with a speed $6\sqrt{2}$ km hr^{-1} in a north-easterly direction. Ship Q is travelling with a speed of $6\sqrt{5}$ km hr^{-1} to intercept P. Let the velocity of Q be $x\vec{i} + y\vec{j}$ km hr^{-1}.



(i) Write down the velocity of P in terms of $\vec{i}$ and $\vec{j}$.

(ii) Find the value of x and the value of y.

(iii) Write down the velocity of Q relative to P in terms of $\vec{i}$ and $\vec{j}$.

(iv) How long does it take Q to intercept P?

Section 4C: Closest Distance

When two bodies are on courses where they don't intercept or collide, there is a time when their paths come closest together. This is known as the closest or least distance.

Solution Strategy for questions involving closest distance:

1) Draw a **detailed diagram** of the system, with the **actual velocities** of the two objects.
2) Put the velocities in $\vec{i}$-$\vec{j}$ form and use $\overrightarrow{v_{ab}} = \overrightarrow{v_a} - \overrightarrow{v_b}$, to **calculate the relative velocity.**
3) Draw **another diagram**, this time showing the **relative velocity** $\overrightarrow{v_{ab}}$. The **closest distance** is the **perpendicular** distance from the relative velocity line $\overrightarrow{v_{ab}}$ to b. This distance, x, can be calculated using trigonometry and geometry.

Example 4C1

Port X is 20 km due north of port Y. Ship *A* leaves port X with velocity $11\vec{i} - 2\vec{j}$ km hr^{-1}, where $\vec{i}$ and $\vec{j}$ are unit vectors in the directions east and north respectively. At the same instant, ship *B* leaves port Y with velocity $5\vec{i} + 6\vec{j}$ km hr^{-1}.

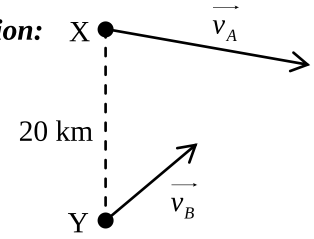
(i) Find the velocity of ship A relative to ship B in terms of $\vec{i}$ and $\vec{j}$.

(ii) Find the magnitude and direction of the velocity of A relative to B, giving the direction to the nearest 0.1°.

(iii) Show on a diagram the initial positions of the two ships as they leave the two ports, and show also on the same diagram the direction in which A is travelling relative to B.

(iv) Calculate the shortest distance between the two ships.

Solution:



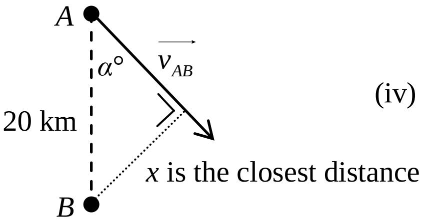
(i) $\overrightarrow{v_{AB}} = \overrightarrow{v_A} - \overrightarrow{v_B} = \left(11\vec{i} - 2\vec{j}\right) - \left(5\vec{i} + 6\vec{j}\right) = \left(11-5\right)\vec{i} - \left(2+6\right)\vec{j}$

$\Rightarrow \overrightarrow{v_{AB}} = 6\vec{i} - 8\vec{j}$ km hr^{-1}

(ii) $\left|\overrightarrow{v_{AB}}\right| = \sqrt{6^2 + \left(-8\right)^2} = 10$ km hr^{-1}

$\tan\alpha = \frac{6}{8} = \frac{3}{4}$, $\Rightarrow \alpha = \tan^{-1}\frac{3}{4} = 36.9^\circ$ east of south.

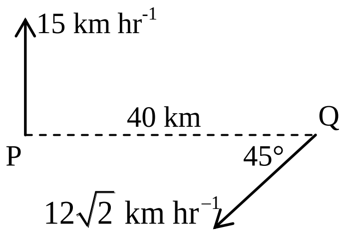
(iii)



(iv) Closest distance $= x = 20\sin\alpha = 20\sin 36.9^\circ = 12$ km.

Example 4C2

Ship P is travelling north with a constant speed of 15 km hr^{-1}. Another ship Q is travelling south-west with a constant speed of $12\sqrt{2}$ km hr^{-1}. At a certain instant, P is positioned 40 km west of Q.



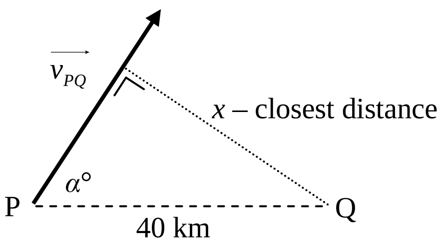
Find:
- (i) the velocity of P in terms of $\vec{i}$ and $\vec{j}$,
- (ii) the velocity of Q in terms of $\vec{i}$ and $\vec{j}$,
- (iii) the velocity of P relative to Q in terms of $\vec{i}$ and $\vec{j}$,
- (iv) the closest distance between the ships in the subsequent motion, to the nearest 0.1 km,
- (v) the distance between the ships one hour after the instant when they were closest together.

Solution: (i) $\overrightarrow{v_P} = 15\vec{j}$ km hr^{-1}

(ii) $\overrightarrow{v_Q} = -12\sqrt{2}\cos 45^\circ\,\vec{i} - 12\sqrt{2}\sin 45^\circ\,\vec{j} = -12\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)\vec{i} - 12\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)\vec{j}$

$\Rightarrow \overrightarrow{v_Q} = -12\vec{i} - 12\vec{j}$ km hr^{-1}

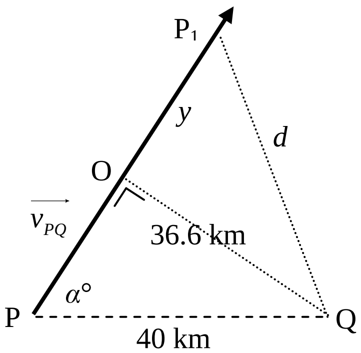
(iii) $\overrightarrow{v_{PQ}} = \overrightarrow{v_P} - \overrightarrow{v_Q} = 15\vec{j} - \left(-12\vec{i} - 12\vec{j}\right) = 12\vec{j} + 27\vec{j}$ km hr^{-1}



(iv) From the direction of $\overrightarrow{v_{PQ}}$: $\tan\alpha = \dfrac{\text{the } \vec{j} \text{ component of } \overrightarrow{v_{PQ}}}{\text{the } \vec{i} \text{ component of } \overrightarrow{v_{PQ}}} = \dfrac{27}{12}$,

$\Rightarrow \alpha = \tan^{-1}\frac{27}{12} = 66.0^\circ$

From the right angled triangle: $x = 40\sin\alpha = 40\sin 66.0^\circ = 36.6$ km.



(v) $\left|\overrightarrow{v_{PQ}}\right| = \sqrt{12^2 + 27^2} = 29.5$ km hr^{-1}

If O is the point where they are closest, then one hour later, P has moved to P_1 along the relative path $\overrightarrow{v_{PQ}}$, and the distance between them is now $\left|QP_1\right| = d$. To find d we must first find $\left|OP_1\right| = y$.

Along $\overrightarrow{v_{PQ}}$, $s = ut$: $\Rightarrow y = 29.5\left(1\right) = 29.5$ km.

Pythagoras: $d^2 = y^2 + 36.6^2$, $\Rightarrow d = \sqrt{29.5^2 + 36.6^2} = 47.0$ km.

Exercise 4C – relative velocity problems involving closest distance

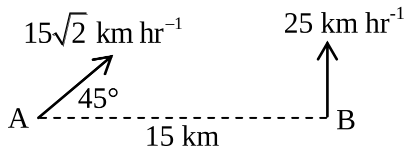
1) The velocity of ship A is $12\vec{i} + \vec{j}$ km hr^{-1} and the velocity of ship B is $8\vec{i} - 2\vec{j}$ km hr^{-1}, where $\vec{i}$ and $\vec{j}$ are unit vectors in the directions east and north respectively.
   - (i) Find the velocity of ship A relative to ship B in terms of $\vec{i}$ and $\vec{j}$.
   - (ii) Find the magnitude and direction of the velocity of A relative to B, giving the direction to the nearest 0.1°.

   At a certain instant, ship A is 20 km due west of ship B.
   - (iii) Show on a diagram the initial positions of the two ships as they leave the two ports, and show also on the same diagram the direction in which A is travelling relative to B.
   - (iv) Calculate the shortest distance between the two ships.

2) The velocity of ship P is $-7\vec{i} - 4\vec{j}$ km hr^{-1} and the velocity of ship Q is $-2\vec{i} + 8\vec{j}$ km hr^{-1}, where $\vec{i}$ and $\vec{j}$ are unit vectors in the directions east and north respectively.
   - (i) Find the velocity of ship Q relative to ship P in terms of $\vec{i}$ and $\vec{j}$.
   - (ii) Find the magnitude and direction of the velocity of P relative to Q, giving the direction to the nearest 0.1°.

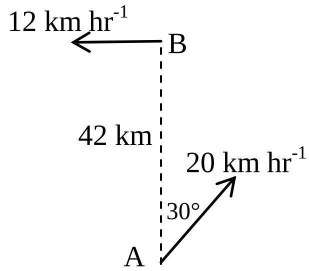
   At a certain instant, ship P is 26 km due north of ship Q.
   - (iii) Show on a diagram the initial positions of the two ships as they leave the two ports, and show also on the same diagram the direction in which P is travelling relative to Q.
   - (iv) Calculate the shortest distance between the two ships.

3) Ship A is travelling north-east at a constant speed of $15\sqrt{2}$ km hr^{-1}.
   Ship B is travelling due north at a constant speed of 25 km h^{-1}.
   At a certain instant B is positioned 15 km due east of A.



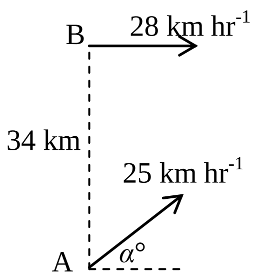
   - (i) Express the velocity of A in terms of $\vec{i}$ and $\vec{j}$.
   - (ii) Express the velocity of B in terms of $\vec{i}$ and $\vec{j}$.
   - (iii) Find the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$.
   - (iv) Calculate the shortest distance between the ships in their subsequent motion.

4) Ship A is positioned 42 km south of ship B.
   Ship A is travelling at a constant speed of 20 km hr^{-1} in the direction N 30° E.
   Ship B is travelling due west at a constant speed of 12 km h^{-1}.



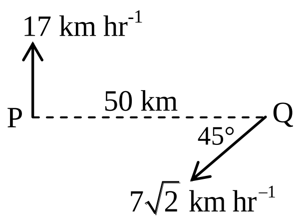
   - (i) Express the velocity of A and the velocity of B in terms of $\vec{i}$ and $\vec{j}$.
   - (ii) Find the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$.
   - (iii) Calculate the shortest distance between the ships in their subsequent motion.

5) Ship A is travelling east α° north with a constant speed of 25 km h^{-1}, where $\tan\alpha = \frac{4}{3}$. Ship B is travelling due east at a constant speed of 28 km h^{-1}. B is positioned 34 km due north of A.



   - (i) Express the velocity of A and the velocity of B in terms of $\vec{i}$ and $\vec{j}$.
   - (ii) Find the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$.
   - (iii) Calculate the shortest distance between the ships in their subsequent motion.
   - (iv) Find the distance between the ships half an hour after the instant that they were closest together.

6) A ship P is travelling north at a constant speed of 17 km h^{-1}. Another ship Q is travelling south-west at a constant speed of $7\sqrt{2}$ km hr^{-1}. At a certain instant, Q is positioned 50 km due east of P.



(i) Express the velocity of P and the velocity of Q in terms of $\vec{i}$ and $\vec{j}$.

(ii) Find the velocity of P relative to Q in terms of $\vec{i}$ and $\vec{j}$.

(iii) Calculate the shortest distance between the ships in their subsequent motion.

(iv) Find the distance between the ships one hour after the instant that they were closest together.

7) The velocity of boat A is $-6\vec{i}+8\vec{j}$ km hr^{-1}. The velocity of boat B is $x\vec{i}+y\vec{j}$ km hr^{-1}.
The velocity of A relative to B is $12\vec{j}$ km hr^{-1}.

(i) Find the value of x and the value of y.

At 2 pm, both A and B sail from the same port P at the same time with velocities as described above.

(ii) After how many hours will they be 24 km apart?

(iii) What is the distance and direction of each boat from the port P when they are 24 km apart?

Section 4D – Overcoming Currents in Rivers

In all of these questions, a moving body (boat, swimmer, etc.) must overcome the current in a river in order to head in the direction it wants to go. There are two special cases to deal with:

(i) For the quickest crossing time: the person must head straight across the river. They will be brought downstream by the current and will end up a certain distance downstream on the opposite bank.

(ii) For the shortest crossing distance: the person must head slightly upstream, in order that they end up going straight across. They will end up directly opposite their starting point on the opposite bank. They must be able to swim faster than the speed of the current in the river for this to be possible.

Solution Strategy for questions involving crossing rivers:

1) **Draw a diagram**. In all relative velocity problems the diagram is very important. We turn the basic formula around to: $\overrightarrow{v_p}=\overrightarrow{v_{pr}}+\overrightarrow{v_r}$, so that we are drawing the diagram as a vector addition, which makes it easier.
2) For questions involving either crossing the river by the **shortest time** or the **shortest distance**, the vector triangle will be right-angled, and so can be easily solved using **Pythagoras' Theorem** and trigonometry.
3) To **calculate the crossing time, divide the width of the river by the $\vec{j}$ component of the velocity.**

Example 4D1

A river is 60 m wide and is flowing with a speed of 1.5 m s^{-1} parallel to the straight banks. The speed of a swimmer in still water is 3 m s^{-1}.

(a) (i) In what direction should the swimmer swim in order to get across the river in the shortest time?
(ii) What is the shortest time it takes the swimmer to swim across the river?
(iii) How far downstream do they end up on the opposite bank?

(b) (i) In what direction should the swimmer swim in order to get across the river by the shortest route?
(ii) How long will it take the swimmer to cross by the shortest route?

Solution: (a)

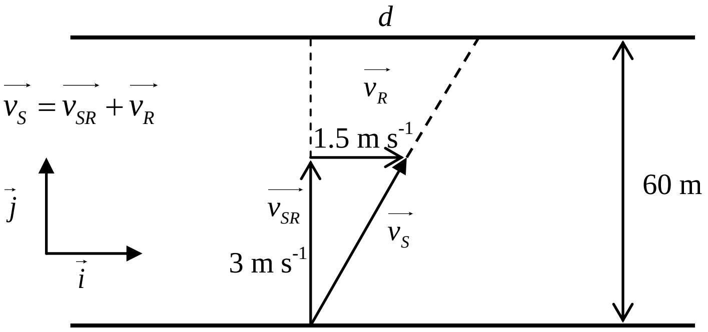
(i) The swimmer should head straight across the river (perpendicular to the bank).

(ii) In the $\vec{j}$ direction:

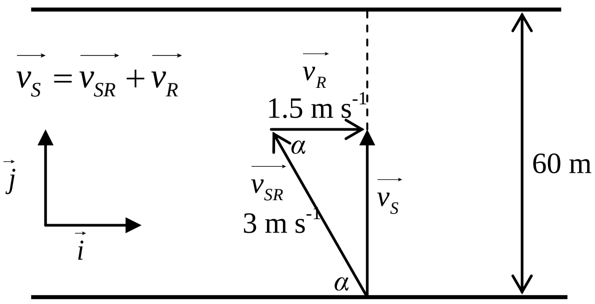
$$t=\frac{\text{width of river}}{\vec{j}\text{ component of velocity}}=\frac{60}{3}=20\text{ s.}$$

(iii) In the $\vec{i}$ direction: $d=1.5\times 20=30$ m.

$\overrightarrow{v_S}=\overrightarrow{v_{SR}}+\overrightarrow{v_R}$



(b)



(i) From the right-angled triangle: $\cos\alpha = \frac{1.5}{3} = \frac{1}{2}$

$\Rightarrow \alpha = \cos^{-1}\frac{1}{2} = 60^\circ$

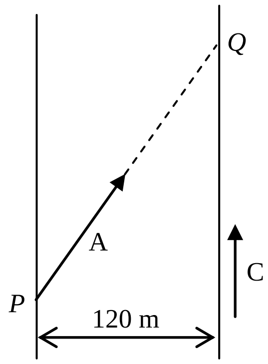
$\Rightarrow$ the swimmer should head at 60° to the upstream bank.

(ii) Using Pythagoras: $\left|\vec{v_S}\right| = \sqrt{3^2 - 1.5^2} = 1.5\sqrt{3}$

$$t = \frac{\text{width of river}}{\vec{j}\text{ component of velocity}} = \frac{60}{1.5\sqrt{3}} = \frac{40}{\sqrt{3}} = 23.1\text{ s.}$$

Example 4D2

A river is 120 m wide and has parallel banks. Boat A departs from point P on its western bank and lands at point Q on its eastern bank. The actual velocity of the boat is $6\vec{i} + 8\vec{j}$ m s^{-1}, where $\vec{i}$ and $\vec{j}$ are unit vectors in the directions east and north respectively.. Cyclist C travels due north at a constant speed of 6 m s^{-1} along the eastern bank of the river. At the same time as boat A is leaving from point P, cyclist C is 180 m south of point Q.



Find: (i) the velocity of C in terms of $\vec{i}$ and $\vec{j}$,

(ii) the velocity of A relative to C in terms of $\vec{i}$ and $\vec{j}$,

(iii) the magnitude and direction of the velocity of A relative to C,

(iv) the time it takes A to cross the river,

(v) $|PQ|$, the distance from P to Q,

(vi) how far cyclist C is from point Q at the moment the boat A reaches point Q.

Solution:

(i) $\vec{v_C} = 6\vec{j}$ m s^{-1}

(ii) $\vec{v_A} = 6\vec{i} + 8\vec{j}$ m s^{-1}

$\vec{v_{AC}} = \vec{v_A} - \vec{v_C} = 6\vec{i} + 2\vec{j}$ m s^{-1}

(iii) $\left|\vec{v_{AC}}\right| = \sqrt{6^2 + 2^2} = \sqrt{40} = 2\sqrt{10} = 6.2$ m s^{-1}

$\tan\alpha = \frac{6}{2} = 3, \quad \Rightarrow \alpha = \tan^{-1}3 = 71.6^\circ$ with the bank.

(iv) In the $\vec{i}$ direction: $s = ut, \quad \Rightarrow t = \frac{s}{u} = \frac{120}{6} = 20$ s.

(v) $\left|\vec{v_A}\right| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$ m s^{-1}

$s = ut: \quad \Rightarrow |PQ| = 10(20) = 200$ m.

(vi) For C, $s = ut = 6(20) = 120$ m

$\Rightarrow$ it is now $180 - 120 = 60$ m from Q.

Exercise 4D – relative velocity problems involving crossing rivers

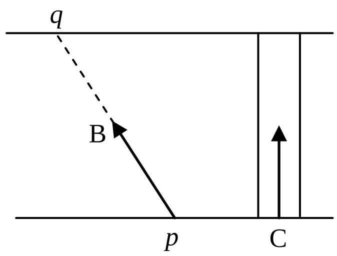
1) A river is 40 m wide and is flowing with a speed of 3 m s^{-1} parallel to the straight banks. The speed of a swimmer in still water is 4 m s^{-1}.

(i) What is the shortest time it takes the swimmer to swim across the river?

(ii) What direction should the swimmer take so as to swim straight across to a point directly opposite? How long will it then take the swimmer to cross to this point?

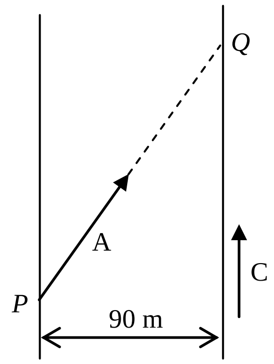
2) A river is 60 m wide and is flowing with a speed of 2 m s^{-1} parallel to the straight banks. The speed of a swimmer in still water is 3 m s^{-1}.
   (i) What is the quickest time it takes the swimmer to swim across the river?
   (ii) Find the time it would take the swimmer to swim across the river by the shortest path.

3) A river is 60 m wide and has parallel banks. A boat B departs from point p on the southern bank and lands at point q on the northern bank. The actual velocity of B is $-3\vec{i}+4\vec{j}$ m s^{-1}. Cyclist C travels due north at a constant speed of 5 m s^{-1} across a straight level bridge which spans the river.



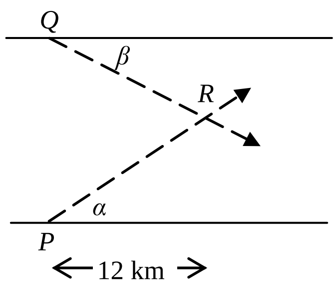
Find: (i) the velocity of C in terms of $\vec{i}$ and $\vec{j}$
   (ii) the velocity of B relative to C in terms of $\vec{i}$ and $\vec{j}$
   (iii) the magnitude and direction of the velocity of B relative to C
   (iv) the time it takes C to cross the river
   (v) how much longer it takes B to cross the river.

4) A river is 90 metres wide and has parallel banks.
Boat B departs from point P on its western bank and lands at point Q on its eastern bank.
The actual velocity of the boat is $6\vec{i}+8\vec{j}$ m s^{-1}.
Cyclist C travels due north at a constant speed of 6 m s^{-1} along the eastern bank of the river.



Find: (i) the velocity of C in terms of $\vec{i}$ and $\vec{j}$,
   (ii) the velocity of B relative to C in terms of $\vec{i}$ and $\vec{j}$,
   (iii) the magnitude and direction of the velocity of B relative to C,
   (iv) the time it takes B to cross the river,
   (v) $|PQ|$, the distance from P to Q.

5) P is on the southern bank of a river.
Q is directly opposite P on the northern bank of the river.
Ship A departs from P at a constant speed of 25 km hr^{-1} and travels in a direction East $\alpha°$ North, where $\tan\alpha = \frac{7}{24}$.
Ship B departs from Q at a constant speed of 39 km hr^{-1} and travels in a direction East $\beta°$ South, where $\tan\beta = \frac{5}{12}$.



Find: (i) the velocity of A in terms of $\vec{i}$ and $\vec{j}$,
   (ii) the velocity of B in terms of $\vec{i}$ and $\vec{j}$,
   (iii) the velocity of A relative to B in terms of $\vec{i}$ and $\vec{j}$.

The paths of A and B intersect at point R, which is 12 km downstream from P and Q.
Find: (iv) the time it takes B to reach R and how much longer it takes A to reach R,
   (v) the width of the river, assuming its banks are parallel.

Answers to Exercises:

Exercise 4A

1) (i) 2 m s^{-1}, (ii) 13 m, (iii) 10 s. 2) (i) 2 m s^{-1}, (ii) 28 m. 3) (i) 35 m s^{-1}, (ii) 22 s.
4) 4 m s^{-1}. 5) 17 m s^{-1}, 61.9° N of W. 6) 31.5 km hr^{-1}, 22.1° S of E.
7) 51.2 km hr^{-1}, 84.4° N of W. 8) 6, -1. 9) 152 km hr^{-1}, 80.5° N of W.
10) (ii) $6\vec{i}-2\vec{j}$ km hr^{-1}, (iii) 10 km hr^{-1}, 53° N of E.

Exercise 4B

1) (i) $16\vec{i}+30\vec{j}$ km hr^{-1}, (ii) $30\vec{j}$ km hr^{-1}, (iii) $16\vec{i}$ km hr^{-1}, (iv) 7 hrs, (v) 238 km, 210 km.

2) (i) $36\vec{j}$ km hr^{-1}, (ii) $-15\vec{i}+36\vec{j}$ km hr^{-1}, (iii) $15\vec{i}$ km hr^{-1}, (iv) 6 hrs, (v) 220 km.

3) (i) $48\vec{i}+14\vec{j}$ km hr^{-1}, (ii) $48\vec{i}$ km hr^{-1}, (iii) $14\vec{j}$ km hr^{-1}, (iv) 4 hrs, (v) 196 km.

4) (ii) $6\vec{i}$ m s^{-1}, (iii) $7.5\sin\alpha\vec{i}+7.5\cos\alpha\vec{j}$ m s^{-1}, (iv) 53.1°, (v) $4.5\vec{j}$ m s^{-1}, (vi) 4 s.

5) (i) $-18\vec{j}$ km hr^{-1}, (ii) $30\cos\alpha\vec{i}-30\sin\alpha\vec{j}$ km hr^{-1}, (iii) 36.9°, (iv) $24\vec{i}$ km hr^{-1}, (v) 15 min.

6) (i) $6\vec{i}+6\vec{j}$ km hr^{-1}, (ii) 6, 12, (iii) $6\vec{j}$ km hr^{-1}, (iv) 3 hrs.

Exercise 4C

1) (i) $4\vec{i}+3\vec{j}$ km hr^{-1}, (ii) 5 km hr^{-1}, 36.9° N of E, (iv) 12 km.

2) (i) $5\vec{i}+12\vec{j}$ km hr^{-1}, (ii) 13 km hr^{-1}, 22.6° E of N, (iv) 10 km.

3) (i) $15\vec{i}+15\vec{j}$ km hr^{-1}, (ii) $25\vec{j}$ km hr^{-1}, (iii) $15\vec{i}-10\vec{j}$ km hr^{-1}, (iv) 8.3 km.

4) (i) $10\vec{i}+10\sqrt{3}\vec{j}$ km hr^{-1}, $-12\vec{i}$ km hr^{-1}, (ii) $22\vec{i}+10\sqrt{3}\vec{j}$ km hr^{-1}, (iii) 33 km.

5) (i) $20\vec{i}+15\vec{j}$ km hr^{-1}, $28\vec{i}$ km hr^{-1}, (ii) $-8\vec{i}+15\vec{j}$ km hr^{-1}, (iii) 16 km, (iv) 18.1 km.

6) (i) $17\vec{j}$ km hr^{-1}, $-7\vec{i}-7\vec{j}$ km hr^{-1}, (ii) $7\vec{i}+24\vec{j}$ km hr^{-1}, (iii) 48 km, (iv) 54.1 km.

7) (i) $-6,-4$, (ii) 2 hrs, (iii) 20 km @ 53.1° N of W, 14.4 km @ 33.7° S of W.

Exercise 4D

1) (i) 10 s, (ii) 41.4°, 15.1 s. 2) (i) 20 s, (ii) 26.8 s. 3) (i) $5\vec{j}$ m s^{-1}, (ii) $-3\vec{i}-\vec{j}$ m s^{-1},

3) (iii) $\sqrt{10}$ m s^{-1}, 18.4° S of W, (iv) 12 s, (v) 3 s. 4) (i) $6\vec{j}$ m s^{-1}, (ii) $6\vec{i}+2\vec{j}$ m s^{-1},

4) (iii) $2\sqrt{10}$ m s^{-1}, 18.4° N of E, (iv) 15 s, (v) 150 m. 5) (i) $24\vec{i}+7\vec{j}$ km hr^{-1},

5) (ii) $36\vec{i}-15\vec{j}$ km hr^{-1}, (iii) $-12\vec{i}+22\vec{j}$ km hr^{-1}, (iv) 20 mins, 10 mins, (v) 8.5 km.